International Journal of Social Statistics - USJ
Volume 02 | Issue 01 | March, 2025

Analytical Representation Technique of Modelling Present Value Function and the Application to Life Table Functions under the Framework of Chebyshev Polynomial

Gbenga Michael Ogungbenle[1], Wipuni Sirisena[2], Ukwu Chukwunenye[3]
[1] Department of Actuarial Science, Faculty of Management Sciences, University of Jos, Nigeria
[2,3]Department of Mathematics, Faculty of Natural Sciences, University of Jos, Nigeria
[1]*moyosiolorun@gmail.com*

Abstract

In life insurance analysis, weighing insured's benefits and contributions which occur over time requires discounting those amounts to present value equivalents. Therefore, the choice of discount rate can be consequential for the valuation of insurance policies. Out of the functions making up the life insurance products, there seems to be no closed form numerical estimates for the interest rate intensity and present value functions. This identified problem may either be in favour of the insured or ortherwise. However, the practice favours the life insurer most in actuarial valuation under the deterministic parsimonious setting. Empirical evidence suggests that new theoretical model advances given the future uncertainty likely suggesting lower long-term rates. This evidence generally supports lowering discount rates under a feasible best guess based on the available financial information. This necessitates deriving a discount rate which can adjust for the fact that benefits are more valuable at present than in the future if policyholders prefer to buy cover now rather than wait or if insurers could be earning a positive return on invested incomes. In this study, the objectives is to develop model for the present value function under the Chebyshev polynomial series framework within the interval of orthogonality and then define some life table structures on the model. From our analytical constructions, as the argument of the polynomial series tends to, we obtain the present value function, which attempts to balance the interests of the policyholders and the life insurers.

Keywords: *Life Insurance, Present Value, Valuation, Policyholders, Financial Information*

1. Introduction

In life insurance, long term policies are standard for instance pension schemes which may consist of periodic payment of contributions until the insured retires followed by periodic benefits until a defined future time. The insured could have paid the initial contribution several years in advance, and this time difference may have a pervasive effect on the valuation of such policy. The value of 1 A life fund's unit measured at some other time is usually governed by the forces of supply and demand, although insurance liability is not easily traded in the most liquid market.

Therefore, time value of money explains why it is sufficient to receive benefit now rather than in the future because of its earning potential and hence accounts for the reason why interest is paid. Consequently, discounting is used to compare benefits and contribution of a scheme or regulation that occurs over time lags. In Bulpitt, T. (n.d.), Canadian Institute of Actuaries (2019a), Canadian Institute of Actuaries (2019b), Insititute and Faculty of Actuaries (2020), discounting the estimates of future cash flows is among the IFRS 17 requirements to reflect the time value of money and investment risk connected with the cash flows assuming these financial risks are not included in the initial estimation of the cash flows. As observed in the paragraph 36 Under IFRS 17, the applicable discount rates for estimating the expected cash flows should reflect the time value of money, the characteristics of cash flows, and the characteristics of insurance contracts' liquidity. It is further required that the effects of the factors affecting the observable market prices used in obtaining the discount rate but not affecting the expected cash flows of insurance contracts should be excluded.

Although, recent studies in actuarial research particularly in life insurance mathematics resulted in proposal of new models for describing and modelling the valuation of mortality, the applicable present value and interest rate modelling for actuarial life valuation are core challenging areas which are yet to be deeply explored under very sophisticated analytical constructions. Therefore, understanding the bevaviour of interest rates under robust analytical models may interest the academic actuary in pricing and reserving insurance products such as life annuities.

Following Cropper and Laibson (1999), Mahbub (2006), Kellison (2009), and Jessop (2018), there are two main rationales for discounting benefits and contributions that occur in the future relative to the present. The future is not certain, and insurers may want to obtain the present benefit value on an insured now rather than later. Secondly, insurance policy holders’ contributions meant to be invested in line with government regulations displace capital that would otherwise be earning a positive return elsewhere in the economy.

Cropper, Freeman, Groom, and Pizer (2014) employ the discount rate in a broad range of financial decisions, including project analysis. Under theoretical modelling, the appropriate discount rate to apply in evaluating a regulation's net costs or benefits depends on whether the regulation basically and directly affects private consumption or private capital. Regulation may directly affect private consumption by raising consumer prices for goods and services. In contrast, regulation may also displace or alter the use of capital in the private sector.

This study aims to shed light on the theoretical methodologies relevant to any assessment of the discount rates in insurance regulatory framework and particular valuation of policy values. The single market interest rate could be an unambiguously correct choice to perform actuarial valuation of benefits in present day terms. However, in life insurance valuation, interest rate are fixed below the market rates because market interest rates often reflect risks associated with capital investments. Consequently, this evidence supports lowering these discount rates with a possible best guess based on the available market information.

Weitzman (1998) and Weitzman (2001) theoretically proved, while Newell and Pizer (2003) and Groom, Koundouri, Panopoulou, and Pantelidis (2007) confirm empirically that discount rate uncertainty could have a marked effect on the net present values. Following Newell and Pizer (2003) and Summers and Zeckhauser (2008), a significant implication from these studies is that a sustained element of uncertainty in the discount rate will result in an effective discount rate that declines over time.

According to Weitzman (1998), Weitzman (2001), Groom et al. (2007) and Gollier (2008), lower discount rates seem to persist over very long period irrespective of whether the estimated investment effects are predominantly measured in private capital.

The essential way to model discount rate uncertainty remains an active area of actuarial research. Gollier (2002) extends the Ramsey framework by considering an extra term that reduces the expected growth rate to account for an uncertain future. Furthermore, Gollier and Hammitt (2014) argue that the impact is to reduce the discount rate, though the effect could be insignificantly small.

Newell and Pizer (2003) employ a model of how long- term interest rates change over time to forecast future discount rates. Their model incorporates some of the basic features of how interest rates change over time and its parameters are estimated based on historical observations of long-term rates. Subsequent work most notably Groom et al. (2007) considers more general models of interest rate dynamics to allow for better forecasts. Specifically, the volatility of interest rates depends on whether rates are currently low or high and variation in the level of persistence over time. In Azevedo (2021) and EIOPA (2021), the discount rate is defined as

$$Discount\,rate = risk\ free\,rate + illiquidity\ premium \tag{1}$$

Following the author's observation, the first step in this approach is to obtain the risk-free rate or yield curve. However, IFRS 17 does not prescribe any technique required to construct a model for the risk-free rate.

Suppose D_e defines the effective rate of discount corresponding to the nominal rate of discount D convertible m times a year. Let A be the maturity amount at the end of 1 year. At D_e effective rate of discount, the present value is given by

$$PV_1 = A\left(1 - D_e\right) \tag{1a}$$

At D nominal rate of discount convertible m times a year, the present value is given by

$$PV_2 = A\left(1 - \frac{D}{m}\right)^m \tag{1b}$$

However, it must be that

$$PV_2 = PV_1 \tag{2}$$

Consequently, we have

$$A\left(1 - \frac{D}{m}\right)^m = A\left(1 - D_e\right) \tag{3}$$

$$\left(1-\frac{D}{m}\right)^{m}=\left(1-D_{e}\right) \quad (4)$$

$$D_{e}=1-\left(1-\frac{D}{m}\right)^{m} \quad (5)$$

Taking the limit of the right hand side, we obtain

$$D_{e}=\lim_{m\to\infty}\left(1-\left(1-\frac{D}{m}\right)^{m}\right) \quad (6)$$

$$D_{e}=1-\lim_{m\to\infty}\left(1-\frac{D}{m}\right)^{m} \quad (7)$$

$$D_{e}=1-\lim_{m\to\infty}\left(1-\frac{D}{m}\right)^{m\times\frac{-D}{-D}} \quad (8)$$

$$D_{e}=1-\lim_{m\to\infty}\left[\left(1-\frac{D}{m}\right)^{\frac{m}{-D}}\right]^{-D} \quad (9)$$

$$D_{e}=1-\lim_{m\to\infty}\left[\left(1-\frac{D}{m}\right)^{\frac{-1}{\frac{D}{m}}}\right]^{-D} \quad (10)$$

Consequently, the effective rate of discount equivalent to the nominal rate of discount convertible continuously is given by

$$D_{e}=1-e^{-D} \quad (10a)$$

The goal of this paper is to leverage on Chebyshev polynomial to build continuous discount and future value functions within the interval of orthogonality $[-1,1]$. The actuarial application of Chebyshev polynomial to the risk free interest rate representation is of great importance to obtain the value of the discount function within a year of insurance transaction. Following Rababah (2003) and Lv and Shen (2017), Chebyshev polynomials have been confirmed to be beneficial as polynomial approximation of a continuous actuarial function with arbitrary precision. It has equal error property as it oscillates between -1 and 1. In order avoid confusion with the complete future life time

$T(x)$, we use C for the Chebyshev polynomial.

The value of 1 unit of a fund at time $s \geq 0$ measured at some time $\tau \geq 0$ is defined by $V(\tau, s)$. If $\tau < s$, then $V(\tau, s)$ is the $(\tau - s)$ discount factor. However, when $\tau > s$, then $V(\tau, s)$ is the $(s - \tau)$ interest factor. Both $V(\tau, s) > 0$ and $V(\tau, s) > 0$. It is believed that interest accumulates frequently say monthly at a rate m and hence we have

$$V(m, 0) = (1+i)^m \tag{11}$$

$$V(0, m) = (1+i)^{-m} \tag{11a}$$

For $\tau, u, s \geq 0$ and from the principle of consistency, we have

$$V(\tau, s) = V(\tau, u) V(u, s) \tag{11b}$$

if $u = \tau$ in (11b)

$$V(\tau, s) = V(\tau, \tau) V(\tau, s) \tag{11c}$$

$$V(\tau, \tau) \equiv 1 \tag{11d}$$

if $\tau = s$ in (11b)

$$V(\tau, \tau) = V(\tau, u) V(u, \tau) \tag{11e}$$

$$V(\tau, u) = \frac{V(\tau, \tau)}{V(u, \tau)} = \frac{1}{V(u, \tau)} \tag{11f}$$

When $\tau = 0$, we have

$$V(0, u) V(u, s) = V(0, s) \tag{11g}$$

2. Materials and Methods

Define

$$C_m : [-1, 1] \to \mathbf{R} \tag{12}$$

At the extreme right of the interval, we generate the future value function; at the extreme left, we produce the discount function. The function $C(s)$ can be approximated using Taylor's series expansion about an arbitrary time s hence

$$C_m(s) = C_m(0) + \frac{(s)}{1!}\frac{d}{ds}C_m(0) + \frac{(s)^2}{2!}\frac{d^2}{ds^2}C_m(0) + \frac{(s)^3}{3!}\frac{d^3}{ds^3}C_m(0) + \ldots + \frac{(s)^k}{k!}\frac{d^k}{ds^k}C_m(0) \quad (13)$$

Given the Chebyshev function defined as $C_m(s) = \cos(m\beta)$ where $s = \cos\beta$

If s is defined within the interval $[-1,1]$ then the interval of β can be taken as $[0,\pi]$. We observe that these ranges are traversed in the opposite sense because $s=-1$ corresponds to $\beta=\pi$ and $s=1$ corresponds to $\beta=0$. Again, $\cos 0\beta = 1$; $\cos 1\beta = \cos\beta$; $\cos 2\beta = 2\cos^2\beta - 1$

$\cos 3\beta = 4\cos^3\beta - 3\cos\beta$; $\cos 4\beta = 8\cos^4\beta - 8\cos^2\beta + 1$;

$$C_0(s) = 1 \quad (14)$$

$$C_1(s) = s = \cos\beta = \cos(-\beta) \quad (15)$$

$$C_2(s) = 2s^2 - 1 \quad (16)$$

$$2s^2 = C_2(s) + 1 \quad (16a)$$

$$s^2 = \frac{C_2(s)+1}{2} = \frac{C_2(s)+C_0(s)}{2} \quad (16b)$$

$$C_3(s) = 4s^3 - 3s \quad (17)$$

$$4s^3 = C_3(s) + 3s \quad (17a)$$

$$4s^3 = C_3(s) + 3C_1(s) \tag{17b}$$

$$s^3 = \frac{C_3(s) + 3C_1(s)}{4} \tag{17c}$$

$$C_4(s) = 8s^4 - 8s^2 + 1 \tag{18}$$

$$8s^4 = C_4(s) + 8s^2 - 1 \tag{18a}$$

$$8s^4 = C_4(s) + \left(\frac{8C_2(s) + 8C_0(s)}{2}\right) - C_0(s) \tag{18b}$$

$$8s^4 = C_4(s) + 4C_2(s) + 4C_0(s) - C_0(s) = C_4(s) + 4C_2(s) + 3C_0(s) \tag{18c}$$

$$s^4 = \frac{C_4(s) + 4C_2(s) + 3C_0(s)}{8} \tag{19}$$

$$C_5(s) = 16s^5 - 20s^3 + 5s \tag{20}$$

$$16s^5 = C_5(s) + 20s^3 - 5s \tag{20a}$$

$$16s^5 = C_5(s) + 20\left[\frac{C_3(s) + 3C_1(s)}{4}\right] - 5C_1(s) \tag{20b}$$

$$16s^5 = C_5(s) + 5\left[C_3(s) + 3C_1(s)\right] - 5C_1(s) \tag{20c}$$

$$16s^5 = C_5(s) + 5C_3(s) + 15C_1(s) - 5C_1(s) \tag{20d}$$

$$16s^5 = C_5(s) + 5C_3(s) + 10C_1(s)$$

(20e)

$$s^5 = \frac{C_5(s) + 5C_3(s) + 10C_1(s)}{16}$$

(21)

$$C_6(s) = 32s^6 - 48s^4 + 18s^2 - 1$$

(22)

$$C_6(s) = 32s^6 - 6C_4(s) - 24C_2(s) - 18C_0(s) + 9C_2(s) + 9C_0(s)$$
$$-C_0(s)$$

(22a)

$$C_6(s) = 32s^6 - 6C_4(s) - 15C_2(s) - 10C_0(s)$$

(22b)

$$32s^6 = C_6(s) + 6C_4(s) + 15C_2(s) + 10C_0(s)$$

(22c)

$$s^6 = \frac{C_6(s) + 6C_4(s) + 15C_2(s) + 10C_0(s)}{32}$$

(22d)

$$C_7(s) = 64s^7 - 112s^5 + 56s^3 - 7s$$

(23)

$$C_7(s) = 64s^7 - 7C_5(s) - 35C_3(s) - 70C_1(s) + 14C_3(s) + 42C_1(s) - 7C_1(s)$$

(23b)

$$C_7(s) = 64s^7 - 7C_5(s) - 21C_3(s) - 35C_1(s)$$

(23c)

$$64s^7 = C_7(s) + 7C_5(s) + 21C_3(s) + 35C_1(s)$$

(23d)

$$s^7 = \frac{C_7(s) + 7C_5(s) + 21C_3(s) + 35C_1(s)}{64}$$

(23e)

$$C_8(s)=128s^8-256s^6+160s^4-32s^2+1 \tag{24}$$

$$C_8(s)=128s^8-8C_6(s)-48C_4(s)-120C_2(s)-80C_0(s)+20C_4(s)$$
$$+80C_2(s)+60C_0(s)-16C_2(s)-16C_0(s)+C_0(s) \tag{24a}$$

$$C_8(s)=128s^8-8C_6(s)-28C_4(s)-56C_2(s)-35C_0(s) \tag{24b}$$

$$128s^8=C_8(s)+8C_6(s)+28C_4(s)+56C_2(s)+35C_0(s) \tag{24c}$$

$$s^8=\frac{C_8(s)+8C_6(s)+28C_4(s)+56C_2(s)+35C_0(s)}{128} \tag{24d}$$

Following the same procedures, we obtain

$$C_9(s)=256s^9-576s^7+432s^5-120s^3+9s \tag{24e}$$

$$C_{10}(s)=512s^{10}-1280s^8+1120s^6-400s^4+50s^2-1 \tag{24f}$$

In general, the Chebyshev polynomial can be defined in terms of the generating function

$$f(s,x)=\frac{1-xs}{1-2xs+s^2}=\sum_{m=0}^{\infty}C_m(x)s^m \tag{24g}$$

$$C_m(x)={}_2F_1\left(-m,m;\frac{1}{2};\frac{1-x}{2}\right) \tag{24h}$$

$$C_m(x)=\left(\frac{m}{2}\right)\sum_{r=0}^{\left\lfloor \frac{m}{2}\right\rfloor}\left[(-1)^r\left(\frac{1}{m-r}\right)\times\left({}^{(m-r)}C_r\right)(2x)^{m-2r}\right];\quad m\geq 1 \tag{24i}$$

Following Abchiche M., and Belbachir H. (2018), Kim, Kim, Jang, Dolgy (2018) and Ricci (2020), the Chebyshev polynomial can be described based on based on Rodrigues equation as follows.

$$C_m\left(x\right)=\frac{\left(-1\right)^m 2^m m!}{\left(2m\right)!}\sqrt{\left(1-x^2\right)}\frac{d^m}{dx^m}\left(1-x^2\right)^{m-0.5}$$

(24j)

The Chebyshev polynomial satisfies the orthogonal properties concerning the weighting function

$$w\left(x\right)=\frac{1}{\sqrt{1-x^2}}$$

(24k)

Employing the inner products of two real valued functions f and g

$$\left\langle f,g\right\rangle=\int_{-1}^{1}w\left(x\right)f\left(x\right)g\left(x\right)dx$$

(24l)

The orthogonality condition requires that

$$\left\langle f,g\right\rangle=0$$

(24m)

$$\left\langle C_i,C_j\right\rangle=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}C_iC_jdx$$

(24n)

$$\left\langle C_i,C_j\right\rangle=\int_{0}^{\pi}\cos i\alpha\sin j\alpha d\alpha$$

(24o)

Letting $x=\cos\alpha$, $dx=-\sin\alpha d\alpha=-\sqrt{1-x^2}d\alpha$, $C_i\left(x\right)=\cos i\alpha$

$$\left\langle C_i,C_j\right\rangle=\frac{1}{2}\int_{0}^{\pi}\left(\cos\left(i+j\right)\alpha+\cos\left(i-j\right)\alpha\right)d\alpha;\quad i\neq j$$

(24p)

$$\left\langle C_i,C_j\right\rangle=\frac{1}{2}\left[\frac{\sin\left(i+j\right)\alpha}{\left(i+j\right)}+\frac{\sin\left(i-j\right)\alpha}{\left(i+j\right)}\right]_{\alpha=0}^{\pi}=0$$

(24q)

$$\left\langle C_i,C_j\right\rangle=0;\quad i\neq j$$

(24r)

Hence, $C_i\left(s\right);\quad i=0,1,2,3,\ldots$ forms an orthogonal polynomial system on $\left[-1,1\right]$ with respect to the weighting function $w\left(x\right)=\frac{1}{\sqrt{1-x^2}}$

Observe that

$$e^{i\beta}=1+i\beta+\frac{(i\beta)^{2}}{2!}+\frac{(i\beta)^{3}}{3!}+\frac{(i\beta)^{4}}{4!}+\frac{(i\beta)^{5}}{5!}\dots \tag{25}$$

Therefore,

$$e^{-i\beta}=1+i\beta-\frac{\beta^{2}}{2!}-i\frac{\beta^{3}}{3!}+\frac{\beta^{4}}{4!}+i\frac{\beta^{5}}{5!}-\dots \tag{26}$$

$$e^{i\beta}=\left(1-\frac{\beta^{2}}{2!}+\frac{\beta^{4}}{4!}-\frac{\beta^{6}}{6!}\right)+i\left(\beta-\frac{\beta^{3}}{3!}+\frac{\beta^{5}}{5!}-\frac{\beta^{7}}{7!}\right)-\dots \tag{27}$$

$$e^{i\beta}=\cos\beta+i\sin\beta \tag{28}$$

$$e^{-i\beta}=\cos\beta-i\sin\beta \tag{29}$$

Therefore,

$$\left(e^{i\beta}+e^{-i\beta}\right)^{m}=e^{im\beta}+{}^{m}C_{1}e^{i(m-2)\beta}+\dots+{}^{m}C_{m-1}e^{-i(m-2)\beta}+e^{-im\beta} \tag{30}$$

$$\left(e^{i\beta}+e^{-i\beta}\right)^{m}=\left(e^{im\beta}+e^{-im\beta}\right)+\left({}^{m}C_{1}e^{i(m-2)\beta}+{}^{m}C_{m-1}e^{-i(m-2)\beta}\right)+{}^{m}C_{2}\left(e^{i(m-4)\beta}+e^{-i(m-4)\beta}\right)+\dots \tag{31}$$

The number of brackets will be $\left\lfloor\frac{m}{2}\right\rfloor+1$ where $\lfloor.\rfloor$ is an integer function. If m is even, the last bracket contains only the one middle term $e^{0\times\beta}=1$
But

$$\left(e^{i\beta}+e^{-i\beta}\right)^{m}=\left(2\cos\beta\right)^{m} \tag{32}$$

$$\left(e^{i\beta}+e^{-i\beta}\right)^{m}=2^{m}\cos^{m}\beta \tag{33}$$

$$2^{m-1}\cos^{m}\beta=\sum_{k=0}^{\left\lfloor\frac{m}{2}\right\rfloor}{}^{m}C_{k}\cos(m-2k)\beta \tag{34}$$

where $\sum.$ denotes that the kth term in the sum must be divided by 2 if m is even and

$$k = \frac{m}{2}$$

Therefore, from the definition of $C_m(s)$

$$s^m = 2^{1-m} \cos^m \beta \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} {}^{m}C_k C_{m-2k}(s) \tag{35}$$

Using equation (15) an $C_m(s) = \cos(m\beta)$ in the trigonometric identity

$$\cos\{(m+1)\zeta\} + \cos\{(m-1)\zeta\} = 2\cos\zeta \cos\{m\zeta\} \tag{36}$$

Then we obtain

$$C_{m+1}(s) + C_{m-1}(s) = 2sC_m(s) \tag{37}$$

$$2C_m(s) = C'_{m+1}(s) - 2sC'_m(s) + C'_{m-1}(s) \tag{37a}$$

$$C_m(s) = \frac{\left[C'_{m+1}(s) - 2sC'_m(s) + C'_{m-1}(s)\right]}{2} \tag{37b}$$

where m is the degree of the polynomial and $m \geq 1$

$$C_m(s) = \sum_{k=0}^{\left\lceil \frac{m}{2} \right\rceil} (-1)^k \, 2^{m-2k-1} \frac{m}{m-k} \binom{m-k}{k} s^{m-2k} \tag{38}$$

and

$$s^m = 2^{1-m} \sum_{k=0}^{\left\lceil \frac{m}{2} \right\rceil} \binom{m}{k} C_{m-2k}(s) \tag{39}$$

where the summation means that we must divide the kth term in the sum by 2 if m is even, $k = \frac{m}{2}$ for instance if $m = 6$. From the above $\left[\frac{m}{2}\right]$ will be the floor (integer) function $\left\lfloor \frac{m}{2} \right\rfloor$ when m is a positive integer and $\left[\frac{m}{2}\right]$ will be the ceiling function $\left\lceil \frac{m}{2} \right\rceil$ if m is odd, for instance $\lfloor 6.5 \rfloor = 6$ and $\lceil 6.5 \rceil = 7$

$$s^6 = 2^{1-6}\sum_{k=0}^{\left[\frac{6}{2}\right]}\binom{6}{k}C_{6-2k}(s) = 2^{-5}\sum_{k=0}^{3}\binom{6}{k}C_{6-2k}(s) =$$

$$2^{-5}\left[\binom{6}{0}C_6(s)+\binom{6}{1}C_{6-2}(s)+\binom{6}{2}C_{6-4}(s)+\frac{1}{2}\binom{6}{3}C_{6-6}(s)\right]$$

(40)

$$s^6 = 2^{-5}\left[C_6(s)+6C_4(s)+15C_2(s)+\frac{1}{2}(20)C_0(s)\right]=$$
$$\frac{\left[C_6(s)+6C_4(s)+15C_2(s)+10C_0(s)\right]}{32}$$

(41)

Furthermore, from the arguments in equation (11b), the one-dimensional discount function is defined as

$$V(u)=V(0,s)$$

(41a)

$$V(\tau,s)=e^{\delta\times(\tau-s)}$$

(41b)

$$V(\tau,u)V(u,s)=e^{\delta\times(\tau-u)}e^{\delta\times(u-s)}=e^{\delta\times(\tau-s)}=V(\tau,s)$$

(41c)

$$\frac{d}{d\tau}V(\tau,s)=\delta e^{\delta\times(\tau-s)}$$

(41d)

$$\left[\frac{d}{d\tau}V(\tau,s)\right]_{s=\tau}=\delta$$

(41e)

$$e^{s\delta}=1+\delta s+\frac{\delta^2s^2}{2!}+\frac{\delta^3s^3}{3!}+\frac{\delta^4s^4}{4!}+\frac{\delta^5s^5}{5!}+\frac{\delta^6s^6}{6!}+\frac{\delta^7s^7}{7!}+\frac{\delta^8s^8}{8!}$$

(42)

Substituting for the powers of s in the above equation (42), we have

$$e^{s\delta}=C_0(s)+\delta C_1(s)+\frac{\delta^2\left[C_2(s)+C_0(s)\right]}{4}+\frac{\delta^3\left[C_3(s)+3C_1(s)\right]}{24}+\frac{\delta^4\left[C_4(s)+4C_2(s)+3C_0(s)\right]}{192}+\frac{\delta^5\left[C_5(s)+5C_3(s)+10C_1(s)\right]}{1920}+\frac{\delta^6\left[C_6(s)+6C_4(s)+15C_2(s)+10C_0(s)\right]}{23040}+\frac{\delta^7\left[C_7(s)+7C_5(s)+21C_3(s)+35C_1(s)\right]}{322560}+\frac{\delta^8\left[C_8(s)+8C_6(s)+28C_4(s)+56C_2(s)+35C_0(s)\right]}{5160960}$$

(43)

$$e^{s\delta}=C_0(s)+\delta C_1(s)+\frac{\delta^2C_2(s)}{4}+\frac{\delta^2C_0(s)}{4}+\frac{\delta^3C_3(s)}{24}+\frac{3\delta^3C_1(s)}{24}+\frac{\delta^4C_4(s)}{192}+\frac{4\delta^4C_2(s)}{192}+\frac{3\delta^4C_0(s)}{192}+\frac{\delta^5C_5(s)}{1920}+\frac{5\delta^5C_3(s)}{1920}+\frac{10\delta^5C_1(s)}{1920}+\frac{\delta^6C_6(s)}{23040}+\frac{6\delta^6C_4(s)}{23040}+\frac{15\delta^6C_2(s)}{23040}+\frac{10\delta^6C_0(s)}{23040}+\frac{\delta^7C_7(s)}{322560}+\frac{7\delta^7C_5(s)}{322560}+\frac{21\delta^7C_3(s)}{322560}+\frac{35\delta^7C_1(s)}{322560}+\frac{\delta^8C_8(s)}{5160960}+\frac{8\delta^8C_6(s)}{5160960}+\frac{28\delta^8C_4(s)}{5160960}+\frac{56\delta^8C_2(s)}{5160960}+\frac{35\delta^8C_0(s)}{5160960}$$

(44)

Collecting the like terms, we have

$$e^{s\delta}=C_0(s)+\frac{\delta^2C_0(s)}{4}+\frac{3\delta^4C_0(s)}{192}+\frac{35\delta^8C_0(s)}{5160960}+\frac{10\delta^6C_0(s)}{23040}+\delta C_1(s)+\frac{10\delta^5C_1(s)}{1920}+\frac{3\delta^3C_1(s)}{24}+\frac{35\delta^7C_1(s)}{322560}+\frac{\delta^2C_2(s)}{4}+\frac{4\delta^4C_2(s)}{192}+\frac{15\delta^6C_2(s)}{23040}+\frac{56\delta^8C_2(s)}{5160960}+\frac{\delta^3C_3(s)}{24}+\frac{5\delta^5C_3(s)}{1920}+\frac{21\delta^7C_3(s)}{322560}+\frac{\delta^4C_4(s)}{192}+\frac{6\delta^6C_4(s)}{23040}+\frac{28\delta^8C_4(s)}{5160960}+\frac{\delta^5C_5(s)}{1920}+\frac{7\delta^7C_5(s)}{322560}+\frac{\delta^6C_6(s)}{23040}+\frac{8\delta^8C_6(s)}{5160960}+\frac{\delta^7C_7(s)}{322560}+\frac{\delta^8C_8(s)}{5160960}$$

(45)

Factoring out the $C_i(s)$ in (45)

$$e^{s\delta}=\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]C_0(s)+\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]C_1(s)+$$
$$\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]C_2(s)$$
$$+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]C_3(s)+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]C_4(s)$$
$$+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]C_5(s)+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]C_6(s)+\frac{\delta^7}{322560}C_7(s)+\frac{\delta^8}{5160960}C_8(s) \quad (46)$$

Substituting the values of $C_i(s)$ in (46), we have

$$e^{s\delta}=\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]+\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+$$
$$\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right](2s^2-1)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right](4s^3-3s)$$
$$+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right](8s^4-8s^2+1)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right](16s^5-20s^3+5s)$$
$$+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right](32s^6-48s^4+18s^2-1)+\frac{\delta^7(64s^7-112s^5+56s^3-7s)}{322560}$$
$$+\frac{\delta^8(128s^8-256s^6+160s^4-32s^2+1)}{5160960} \quad (47)$$

3. Result and Discussion

In (47), let $s=1$ so that we can obtain the future value function

$$
\begin{aligned}
e^{\delta} &= \left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]+\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]+ \\
&\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right] \\
&+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right] \\
&+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]+\frac{\delta^7}{322560} \\
&+\frac{\delta^8}{5160960}
\end{aligned}
\tag{48}
$$

The effective rate of interest i can be obtained from the exact equation

$$\delta=\log_e(1+i) \tag{48a}$$

Using (48a), we have

$$
\begin{aligned}
e^{\delta}-1 &= \left[\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]+\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]+ \\
&\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right] \\
&+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right] \\
&+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]+\frac{\delta^7}{322560} \\
&+\frac{\delta^8}{5160960}
\end{aligned}
\tag{49}
$$

$$\frac{\delta}{e^{\delta}-1}=\frac{\delta}{\left\{\begin{array}{l}\left[\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]+\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]+\\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]+\frac{\delta^7}{322560}\\ +\frac{\delta^8}{5160960}\end{array}\right\}}$$

(50)

Equation (50) therefore defines a relationship between Chebyshev polynomial and Bernoulli Power series.

$$f(\delta)=\frac{\delta}{e^{\delta}-1}=\sum_{m=0}^{\infty}B_m\frac{\delta^m}{m!}$$

(51)

Then we need to obtain the Bernoulli numbers B_m using the following techniques.

$$\lim_{\delta\to 0} f(\delta)=\lim_{\delta\to 0}\frac{\delta}{e^{\delta}-1}$$

(52)

$$\lim_{\delta\to 0} f(\delta)=\lim_{\delta\to 0}\frac{1}{e^{\delta}}$$

(53)

$$\lim_{\delta\to 0} f(\delta)=\lim_{\delta\to 0}\frac{1}{e^{\delta}}=1$$

(54)

$$f'(\delta)=\frac{e^{\delta}-1-\delta e^{\delta}}{\left(e^{\delta}-1\right)^2}$$

(55)

$$\lim_{\delta\to 0} f'(\delta)=\lim_{\delta\to 0}\frac{e^{\delta}-1-\delta e^{\delta}}{\left(e^{\delta}-1\right)^2}$$

(56)

$$\lim_{\delta\to0} f'(\delta)=\lim_{\delta\to0}\frac{e^{\delta}-e^{\delta}-\delta e^{\delta}}{2e^{\delta}\left(e^{\delta}-1\right)}$$

(57)

$$\lim_{\delta\to0} f'(\delta)=\lim_{\delta\to0}\frac{-\delta}{2\left(e^{\delta}-1\right)}$$

(58)

$$\lim_{\delta\to0} f'(\delta)=\lim_{\delta\to0}\frac{-1}{2\left(e^{\delta}\right)}$$

(59)

$$\lim_{\delta\to0} f'(\delta)=\frac{-1}{2}$$

(60)

$$f^{(2)}(\delta)=\left(e^{\delta}-e^{\delta}-\delta e^{\delta}\right)\left(e^{\delta}-1\right)^{-2}+\left(e^{\delta}-1-\delta e^{\delta}\right)(-2)e^{\delta}\left(e^{\delta}-1\right)^{-3}$$

(61)

$$\lim_{\delta\to0} f^{(2)}(\delta)=\lim_{\delta\to0}\frac{\left(-\delta e^{\delta}\right)}{\left(e^{\delta}-1\right)^{2}}+\lim_{\delta\to0}\frac{\left(e^{\delta}-1-\delta e^{\delta}\right)(-2)e^{\delta}}{\left(e^{\delta}-1\right)^{3}}$$

(62)

$$\lim_{\delta\to0} f^{(2)}(\delta)=\lim_{\delta\to0}\frac{-e^{\delta}-\delta e^{\delta}}{2e^{\delta}\left(e^{\delta}-1\right)}+(-2)\lim_{\delta\to0}\frac{e^{\delta}\left(e^{\delta}-1-\delta e^{\delta}\right)+e^{\delta}\left(e^{\delta}-e^{\delta}-\delta e^{\delta}\right)}{3e^{\delta}\left(e^{\delta}-1\right)^{2}}$$

(63)

$$\lim_{\delta\to0} f^{(2)}(\delta)=\lim_{\delta\to0}\frac{-1-\delta}{2\left(e^{\delta}-1\right)}+(-2)\lim_{\delta\to0}\frac{\left(e^{\delta}-1-\delta e^{\delta}\right)+\left(-\delta e^{\delta}\right)}{3\left(e^{\delta}-1\right)^{2}}$$

(64)

$$\lim_{\delta\to0} f^{(2)}(\delta)=\lim_{\delta\to0}\frac{-1}{2\left(e^{\delta}\right)}+(-2)\lim_{\delta\to0}\frac{\left(e^{\delta}-e^{\delta}-\delta e^{\delta}\right)-e^{\delta}-\delta e^{\delta}}{6e^{\delta}\left(e^{\delta}-1\right)}$$

(65)

$$\lim_{\delta\to 0} f^{(2)}(\delta) = \frac{-1}{2} + (-2)\lim_{\delta\to 0}\frac{(-\delta)-1-\delta}{6\left(e^{\delta}-1\right)}$$ (66)

$$\lim_{\delta\to 0} f^{(2)}(\delta) = \frac{-1}{2} + (-2)\lim_{\delta\to 0}\frac{(-2\delta)-1}{6\left(e^{\delta}-1\right)}$$ (67)

$$\lim_{\delta\to 0} f^{(2)}(\delta) = \frac{-1}{2} + (-2)\lim_{\delta\to 0}\frac{(-2)}{6\left(e^{\delta}\right)}$$ (68)

$$\lim_{\delta\to 0} f^{(2)}(\delta) = \frac{-1}{2} + \frac{4}{6} = \frac{1}{6}$$ (69)

$$B_m = \sum_{k=0}^{m}\frac{1}{k+1}\sum_{r=0}^{k}(-1)^r \times \left({}^{k}C_r\right) \times r^m$$ (70)

B_m is the mth derivative of $\dfrac{\delta}{e^{\delta}-1}$ evaluated at $\delta = 0$

$$B_m = \frac{d^m}{d\delta^m}\left(\frac{\delta}{e^{\delta}-1}\right)\Bigg|_{\delta=0}$$ (71)

Let

$$e^U = \left(1-\left(1-e^U\right)\right)$$ (72)

Observe that

$$\log_e \delta = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(\delta-1)^k}{k}$$ (73)

$$\log_e(1-\delta) = -\sum_{k=1}^{\infty}\frac{\delta^k}{k}$$ (74)

$$\delta = \sum_{k=1}^{\infty} \frac{\left(1-e^{\delta}\right)^{k}}{k} \tag{75}$$

$$\frac{\delta}{e^{\delta}-1} = \sum_{k=1}^{\infty} \frac{\left(1-e^{\delta}\right)^{k-1}}{k} \tag{76}$$

Replacing k by $k+1$, we have

$$\frac{\delta}{e^{\delta}-1} = \sum_{k=0}^{\infty} \frac{\left(1-e^{\delta}\right)^{k}}{k+1} \tag{77}$$

Substitute (77) in (71), we have

$$B_m = \frac{d^m}{d\delta^m} \sum_{k=0}^{\infty} \frac{\left(1-e^{\delta}\right)^{k}}{k+1} \Bigg|_{\delta=0} \tag{78}$$

$$B_m = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right) \frac{d^m}{d\delta^m} \left(1-e^{\delta}\right)^{k} \Bigg|_{\delta=0} \tag{79}$$

Therefore,

$$\frac{\delta}{e^{\delta}-1} = \sum_{m=0}^{\infty} \lim_{\delta \to 0} \left(f^{(m)}(\delta)\right) \frac{\delta^m}{m!} \tag{80}$$

$$\frac{\delta}{e^{\delta}-1} = 1 - \frac{1}{2}\delta + \frac{1}{6}\frac{\delta^2}{2!} - \frac{1}{30}\frac{\delta^4}{4!} + \frac{1}{42}\frac{\delta^6}{6!} + \ldots \tag{81}$$

$$\frac{\delta}{e^{\delta}-1} = 1 - \frac{1}{2}\delta + \frac{\delta^2}{12} - \frac{\delta^4}{720} + \frac{\delta^6}{30240} + \ldots \tag{82}$$

Therefore, the actuarial discount function is obtained by replacing s by $-s$ In (47), we obtain the discount function

$$e^{-s\delta} = 1 - \delta s + \frac{\delta^2 s^2}{2!} - \frac{\delta^3 s^3}{3!} + \frac{\delta^4 s^4}{4!} - \frac{\delta^5 s^5}{5!} + \frac{\delta^6 s^6}{6!} - \frac{\delta^7 s^7}{7!} + \frac{\delta^8 s^8}{8!} \tag{83}$$

The discount function is obtained by replacing s by $-s$ in equation (47) becomes

$$e^{-s\delta}=\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+$$
$$\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2s^2-1\right)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4s^3+3s\right)$$
$$+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8s^4-8s^2+1\right)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16s^5+20s^3-5s\right)$$
$$+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32s^6-48s^4+18s^2-1\right)+\frac{\delta^7\left(-64s^7+112s^5-56s^3+7s\right)}{322560}$$
$$+\frac{\delta^8\left(128s^8-256s^6+160s^4-32s^2+1\right)}{5160960}$$

(84)

when $s=1$ In equation (84), then we obtain the present value function.

$$v=\frac{1}{1+i}=e^{-\delta}=\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]+$$
$$\left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]-\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]+\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]$$
$$-\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]+\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]-\frac{\delta^7}{322560}+\frac{232\delta^8}{5160960}$$

(85)

We can now define some life table structures using equation (85) as follows.
Recall that $f_{T_x}(s)=\mu_{x+s}\times{}_s\,p_x$ Is the death density of the random lifetime T_x, then we write

$$\mathbf{E}\left(e^{-\delta T_x}\right)=\int_0^\infty e^{-\delta s} f_{T_x}(s)\,ds$$

(85a)

Following, Souza (2019), the actuarial present value of a whole life annuity benefit of 1 is defined as

$$\bar{a}_x=\int_0^\infty e^{-\delta s}\left({}_s\,p_x\right)ds$$

(85b)

$$\bar{a}_x = \int_0^{\Omega-x} \left\{ \begin{array}{l} \left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+ \\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right](2s^2-1)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right](-4s^3+3s) \\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right](8s^4-8s^2+1)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right](-16s^5+20s^3-5s) \\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right](32s^6-48s^4+18s^2-1)+\frac{\delta^7(-64s^7+112s^5-56s^3+7s)}{322560} \\ +\frac{\delta^8(128s^8-256s^6+160s^4-32s^2+1)}{5160960} \end{array} \right\} ({}_s p_x)\, ds$$

(85b)

The actuarial present value of the whole life insurance benefits of 1 is defined as

$$\bar{A}_x = \int_0^{\infty} e^{-\delta s} \mu_{x+s} ({}_s p_x)\, ds$$

(86)

$$\bar{A}_x = \int_0^{\Omega-x} \left\{ \begin{array}{l} \left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+ \\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right](2s^2-1)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right](-4s^3+3s) \\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right](8s^4-8s^2+1)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right](-16s^5+20s^3-5s) \\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right](32s^6-48s^4+18s^2-1)+\frac{\delta^7(-64s^7+112s^5-56s^3+7s)}{322560} \\ +\frac{\delta^8(128s^8-256s^6+160s^4-32s^2+1)}{5160960} \end{array} \right\} \mu_{x+s} ({}_s p_x)\, ds$$

(87)

As $\delta \to 0$ in (86), we obtain,

$$\bar{e}_x = \int_0^{\Omega-x} ({}_s p_x)\, ds = \int_0^{\Omega-x} e^{-\int_0^s \mu_{x+\theta} d\theta}\, ds$$

(88)

The discrete whole life annuity is given as

$$\ddot{a}_x=\sum_{s=0}^{\Omega-x-1}\left\{\begin{array}{l}\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+\\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2s^2-1\right)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4s^3+3s\right)\\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8s^4-8s^2+1\right)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16s^5+20s^3-5s\right)\\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32s^6-48s^4+18s^2-1\right)+\frac{\delta^7\left(-64s^7+112s^5-56s^3+7s\right)}{322560}\\ +\frac{\delta^8\left(128s^8-256s^6+160s^4-32s^2+1\right)}{5160960}\end{array}\right\}\left({}_s p_x\right) \tag{88a}$$

The discrete whole life annuity is given by

$$A_x=\sum_{s=0}^{\Omega-x-1}v^{s+1}{}_{s|1}q_x \tag{89}$$

$$A_x=\sum_{s=0}^{\Omega-x-1}\left\{\begin{array}{l}\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right](s+1)+\\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2(s+1)^2-1\right)\\ +\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4(s+1)^3+3(s+1)\right)\\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8(s+1)^4-8(s+1)^2+1\right)\\ +\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16(s+1)^5+20(s+1)^3-5(s+1)\right)\\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32(s+1)^6-48(s+1)^4+18(s+1)^2-1\right)\\ +\frac{\delta^7\left(-64(s+1)^7+112(s+1)^5-56(s+1)^3+7(s+1)\right)}{322560}\\ +\frac{\delta^8\left(128(s+1)^8-256(s+1)^6+160(s+1)^4-32(s+1)^2+1\right)}{5160960}\end{array}\right\}\left({}_{k|1}q_x\right) \tag{90}$$

In fixing the premium rates, the actuary should observe the technical basis embodied in assumptions (Anggraeni, Rahmadani, Utama & Handayani. 2023). However, in Cruz (2019), at the policy's inception, the basis observed at this material time is defined as first-order basis. For the life insurer to remain solvent, premiums are expected to cover benefits paid to the insured and other expenses incurred on the policy. Consequently, the benchmark is to compute premiums in line with the equivalence principle. The random variable of interest for the premium computation is the future loss random variable $\ell_s;\quad s>0$. The future loss random variable is the difference between the present value of future benefits and expenses and the present value of future premium. At the outset of the contract, $s=0$. According to the equivalence principle, the premium is computed as

$$E\left(\ell_s\right)=\mathbf{E}\left(Present\,Value\,of\;\;future\,outgo\right)-\mathbf{E}\left(Present\,Value\,of\;\;future\,income\right)=0 \tag{91}$$

Under this principle, the premium and benefits will balance out on the average. Therefore, for a whole life policy incepted on a life aged x with a sum assured of b payable on the death of the insured, the premium π_x is to be paid by the insured as a life continuous annuity. Under the equivalence principle, the premium is computed as $\pi_x=b\times\dfrac{\bar{A}_x}{\ddot{a}_x}$

At the inception of the policy, the insured and the insurer will know the premium amount π_x and the death benefits amount b

$$\pi_x=\frac{b\displaystyle\int_0^{\Omega-x}\left\{\begin{array}{l}\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+\\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2s^2-1\right)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4s^3+3s\right)\\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8s^4-8s^2+1\right)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16s^5+20s^3-5s\right)\\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32s^6-48s^4+18s^2-1\right)+\frac{\delta^7\left(-64s^7+112s^5-56s^3+7s\right)}{322560}\\ +\frac{\delta^8\left(128s^8-256s^6+160s^4-32s^2+1\right)}{5160960}\end{array}\right\}\mu_{x+s}\left({}_sp_x\right)ds}{\displaystyle\sum_{s=0}^{\Omega-x-1}\left\{\begin{array}{l}\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+\\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2s^2-1\right)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4s^3+3s\right)\\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8s^4-8s^2+1\right)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16s^5+20s^3-5s\right)\\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32s^6-48s^4+18s^2-1\right)+\frac{\delta^7\left(-64s^7+112s^5-56s^3+7s\right)}{322560}\\ +\frac{\delta^8\left(128s^8-256s^6+160s^4-32s^2+1\right)}{5160960}\end{array}\right\}\left({}_sp_x\right)} \tag{92}$$

As a hedging strategy, the m-year pure endowment used to offset the losses from the m-year term insurance issued to a life aged x Before any mortality shock is given by

$$A_{x:\overset{\;1}{\overline{m|}}} = v^m {}_m p_x \tag{93}$$

Let

$$\mu_{x+t} = \frac{f_{T_x}(t)}{S_{T_x}(t)} = \frac{f_{T_x}(t)}{1-F_{T_x}(t)} \tag{94}$$

$$\mu_{x+t} = \frac{f_{T_x}(t)}{1-\int_0^t f_{T_x}(\theta)d\theta} \tag{95}$$

$$\int_0^t \mu_{x+\theta}d\theta = \int_0^t \left(\frac{f_{T_x}(\theta)}{1-\int_0^t f_{T_x}(\theta)d\theta} \right) d\theta = -\log_e\left(1-\int_0^t f_{T_x}(\theta)d\theta\right) \tag{96}$$

$$\log_e\left(1-\int_0^t f_{T_x}(\theta)d\theta\right) = -\int_0^t \mu_{x+\theta}d\theta \tag{97}$$

$$1-\int_0^t f_{T_x}(\theta)d\theta = e^{-\int_0^t \mu_{x+\theta}d\theta} \tag{98}$$

From (94),

$$S_{T_x}(t) = \frac{f_{T_x}(t)}{\mu_{x+t}} \tag{99}$$

Let

$$f(t) = \mu_{x+t}e^{-\int_0^t \mu_{x+\theta}d\theta} \tag{100}$$

Then

$$S_{T_x}(t) = \frac{\mu_{x+t}e^{-\int_0^t \mu_{x+\theta}d\theta}}{\mu_{x+t}} = e^{-\int_0^t \mu_{x+\theta}d\theta}$$

(101)

$$A_{x:\overline{s}|}^{\;1}$$

$$=\begin{Bmatrix} e^{-s\delta}=\left[1+\frac{\delta^2}{4}+\frac{3\delta^4}{192}+\frac{35\delta^8}{5160960}+\frac{10\delta^6}{23040}\right]-\left[\delta+\frac{10\delta^5}{1920}+\frac{3\delta^3}{24}+\frac{35\delta^7}{322560}\right]s+ \\ \left[\frac{\delta^2}{4}+\frac{4\delta^4}{192}+\frac{15\delta^6}{23040}+\frac{56\delta^8}{5160960}\right]\left(2s^2-1\right)+\left[\frac{\delta^3}{24}+\frac{5\delta^5}{1920}+\frac{21\delta^7}{322560}\right]\left(-4s^3+3s\right) \\ +\left[\frac{\delta^4}{192}+\frac{6\delta^6}{23040}+\frac{28\delta^8}{5160960}\right]\left(8s^4-8s^2+1\right)+\left[\frac{\delta^5}{1920}+\frac{7\delta^7}{322560}\right]\left(-16s^5+20s^3-5s\right) \\ +\left[\frac{\delta^6}{23040}+\frac{8\delta^8}{5160960}\right]\left(32s^6-48s^4+18s^2-1\right)+\frac{\delta^7\left(-64s^7+112s^5-56s^3+7s\right)}{322560} \\ +\frac{\delta^8\left(128s^8-256s^6+160s^4-32s^2+1\right)}{5160960} \end{Bmatrix}\left({}_s p_x\right)$$

(102)

where

$${}_s p_x = e^{-\int_0^s \mu_{x+\theta}d\theta} \qquad (103)$$

The equations in (84) are the only principal equations used in computing all mortality functions.

4. Conclusion

This paper attempts to model the continuous present and future value function with more sophisticated analytical derivations and apply it to define some life table functions. Life table functions such as the force of mortality, survival function used in actuarial valuation have been estimated using polynomials except the present value function. The modelling of the present value problem and applications in the actuarial life table under the Chebyshev polynomial framework has been most recently developed. The method used in this paper is an alternative to the commonly used method for present value computations but is far more advanced. The strong point of this technique in estimating the present value function is the novelty involved. Using the power series based on orthogonal Chebyshev polynomial, the necessity to compute and implement a robust discount function is guaranteed.

References

Abchiche M., and Belbachir H. (2018). Generalized Chebyshev polynomials. *De Gruyter,* 38, 78-89. Doi:10.7151/dmgaa.1278

Anggraeni A.S., Rahmadani S., Utama R.C., Handayani V.A. (2023). Calculate Indonesian pension funds using group self anuitization method and makeham mortality law. *Jurnal Ilmu Keuangan dan Perbankan,* 12(2), 269-280.

Azevedo G.D.M. (2021). Risk consulting in insurance-IFRS 17, Masters thesis, Lisbon School of Economics and Management, Universidade de Lisboa.

Bulpitt, T. (n.d.). *IFRS 17: Future of Discount Rates.* Retrieved from Institute and Faculty of Actuaries:https://www.actuaries.org.uk/system/files/field/document/IFRS17%20IFoA%20 WG%20-%20discount%20rates%20liquidity%20premium.pdf

Canadian Institute of Actuaries (2019a). *IFRS 17: Estimates of Future Cash Flows for Life and Health Insurance Contracts.* Retrieved from Canadian Institute of Actuaries.

Canadian Institute of Actuaries (2019b). *IFRS 17 Risk Adjustment for Non-Financial Risk for Life and Health Insurance Contracts.* Retrieved from Canadian Institute of Actuaries: https://www.cia-ica.ca/docs/default-source/2019/219081e.pdf

Cropper, M.L., Freeman, M.C., Groom, B. and Pizer W.A. (2014). Declining discount rates, *American Economic Review, Papers and Proceedings* 104(5), 538-543.

Cropper, M. and Laibson D. (1999). The implications of hyperbolic discounting for project evaluation, in *Discounting and Intergenerational Equity*, ed. P.R. Portney and J. P. Weyant. Washington, D.C.: Resources for the Future.

Cruz, M. (2019). *Evaluation of the Technical Provisions of Insurance Contracts under*

IFRS 17. ISEG.

EIOPA (2021). *Risk-free interest rate term structures.* Retrieved from EIOPA: https://www.eiopa.europa.eu/tools-and-data/risk-free-interest-rate-term-structures_en

Gollier, C. (2002). Discounting an uncertain future. *Journal of Public Economics* 85, 149-166.

Gollier, C. (2008). Discounting with fat-tailed economic growth. *Journal of Risk and Uncertainty* 37(2): 171-186.

Gollier, C., and Hammitt, J. (2014. The long-run discount rate controversy. *Annual Review of Resource Economics* 6(1), 273-295.

Groom, B., P. Koundouri, Panopoulou, E. and Pantelidis. T. (2007). Discounting the distant future: How much does model selection affect the certainty equivalent rate? *Journal of Applied Econometrics, 22*, 641–56.

Institute and Faculty of Actuaries (2020). *IFRS 17 - Future of Discount Rates Working Party.* Retrieved from Institute and Faculty of Actuaries: https://www.actuaries.org.uk/system/files/field/document/IFRS%2017%20Publi cation_Final.pdf

Jessop, N. (2018). *Permitted approaches for constructing IFRS 17 Discount Rates*. Retrieved from Moody's Analytics: https://www.moodysanalytics.com/articles/2018/permitted-approaches-for-constructing-ifrs17-discount-rates

Kellison S.G. (2009). *The theory of interest*. McGraw Hill, New York.

Kim, T., Kim, D.S., Jang, C.J., Dolgy D.V. (2018). Representation by Chebyshev polynomials for sums of finite products of Chebyshev polynomials. Symmetry, 10, 742, 1-14. Doi:10.3390/sym10120742

Lv, X., and Shen S. (2017). On Chebyshev polynomials and their applications. *Advances in Differential Equations,* 343, 1-6. Doi: 10.1186/s13662-017-1387-8

Mahbub P. (2006). Time value of money: Application and rationality-An approach using differential equations and definite integrals. *MPRA*, 1-10. https://mpra.ib.uni-muenchen.de/10794/

Newell, R., and Pizer, W. (2003). Discounting the distant future: How much do uncertain rates

Increase valuations? *Journal of Environmental Economics and Management, 46* (1), 52–71.

Rababah A. (2003). Transformation of Chebyshev-Bernstein polynomial basis. *Computational Methods in Applied Mathematics,* 3(4), 608-622.

Ricci P.E. (2020). Chebyshev polynomials, Rhodonea curves, and pseudo-Chebyshev functions. A survey. *Growth and Form,* 1(1), 20-32. Doi: https://doi.org/10.2991/gaf.k.200124.005

Souza, F.C. (2019). Upper and lower bounds for annuities and life insurance from incomplete mortality data. *Revista Contabilidade & Financas,* 30(80). 282-291. doi: 10.1590/1808 057x201807320

Summers, L. and Zeckhauser, R. (2008). Policymaking for Posterity, Harvard Kennedy School Faculty Research Working Papers Series.

Weitzman, M. L. (1998). Why the far-distant future should be discounted at its lowest possible rate. *Journal of Environmental Economics and Management, 36*, 201–8.

Weitzman, M. L. (2001). Gamma discounting, *American Economic Review 91*(1), 260-271.