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A Comparative Mathematical Study of the Relationship Between Marginal Social Cost and Pigouvian Tax in the Presence of Commodity and Wage Taxes: Putting Ramsey theorem into Practice

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Abstract

The aim of this paper is to examine the relationship between Pigouvian tax and marginal social cost in the presence of distortionary taxes such as commodity and wage taxes in a Ramsey setting. The Ramsey theory highlights the amount of tax required to raise a given revenue for the government which also maximizes household utility. Previous research in this regard has been carried out either under homogeneous household preferences or constant marginal social cost. In this paper we go further by analyzing the relationship between Pigouvian tax and marginal social cost in the presence of commodity taxes when households have heterogeneous preferences as opposed to being assumed homogeneous. In addition, we also consider the relationship between Pigouvian tax and marginal social cost in the presence of wage tax when the marginal social cost is considered as a variable depending on Pigouvian tax as opposed to being considered a constant in previous literature. The results indicate that the Pigouvian tax in the presence of wage tax is higher when the marginal social cost was considered a variable as opposed to a constant. Under certain conditions, in the presence of commodity taxes it was observed that the value of the Pigouvian tax is higher when households have heterogeneous preferences as opposed to homogeneous preferences. The mathematical models used in this study enable to see the factors, such as homogeneity/heterogeneity of household

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preferences and marginal social cost assumed as a variable as opposed to a constant, that impact the dynamics in determining the optimal Pigouvian tax.

Keywords: *Ramsey Theory, Pigou Tax, Marginal Social Cost, Lagrange Multiplier*

Introduction

Sustainability in the economy is a multi-faceted discussion which is deemed highly relevant, in the current global context, by many experts. One of the key discussions pertaining to sustainability in the economy is the conversation revolving around environmental taxation. To explain the importance of environmental taxation and social cost, consider the example of the infamous case pertaining to DuPont in the 1960's. According to Rich (2016), in an article published in the New York Times magazine, DuPont factories manufactured Teflon nonstick pans since 1960's using a chemical known as PFOA. These chemicals were eventually disposed to nearby rivers and through gas emissions. The water and air were contaminated with PFOA chemicals. DuPont factory workers and people in neighborhoods were affected in terms of health, causing significant rise in cancer (Rich, 2016). The example of DuPont shows that the actions of a company are resulting in society bearing a loss i.e. a social cost. The activity causing this loss, in economics terms, is known as a Negative Externality (Ahuja, 2016). Negative externality causes an external cost on the third party (party which is outside the two main parties, the consumer and the producer) by a producer due to the production of harmful goods which may have negative implications on the third party (Ahuja, 2016). This implies that the producer is not considering the true cost in the manufacturing process. The production cost should not only consider cost pertaining to labor cost, fixed cost, raw material cost, transaction cost etc. These are all private costs. It should also include the external cost such as health-related cost of the people due to air pollution, contamination of rivers and the cost of cleaning the river, etc. caused by the production of such harmful goods. If this external cost is not included, the product price does not reflect the actual social cost and hence results in an overproduction. This leads to an economic concept known as Market Failure. Market Failure is where there exists an inefficient allocation of goods and services.

Various measures have been discussed and are undertaken to address this issue of negative externality caused by overproduction due to market failure. The aim of this research focuses upon examining the relationship between Pigouvian tax and marginal social cost in the presence of distortionary taxes such as commodity and wage taxes in a Ramsey setting (It should also be noted that the administrative cost pertaining to implementing a Pigouvian tax was not the objective of this research, and it is another area where extensive research has been carried out. For example the paper published by Polinsky et al. (1981) looks into how the Pigouvian tax should be adjusted to reflect administrative costs and is a good starting point but is not related to this research). Correcting externalities by equating the marginal social cost to Pigouvian tax has been a long-discussed area (especially when in the context of distortionary taxes). Marginal social cost is simply the change in social cost due to a unit change in production or consumption of a good. Equating the Pigouvian tax to the marginal social cost, i.e. imposing lump sum taxes to eliminate the social cost, is named as Theory of First Best. It is a policy where the resulting equilibrium would be called "First Best".

There would be no market distortions considered in the first best case. However, the real world consists of market distortions mainly caused by other government taxes. In such a scenario obtaining the equilibrium position is called "Theory of Second Best". That is if a constraint is introduced into the general equilibrium system which prevents attaining one of the pareto conditions, then an optimum situation can be achieved only by departing from all other pareto conditions (Lipsey et al., 1956). In this regard, equating the two will not always eliminate the negative externality. In this regard, an important inquiry arises as to what the actual relationship should be when considering the requirement of government revenue (Ramsey, 1927) and simultaneously the utility maximization of households. An in-depth mathematical analysis of the optimal Pigouvian tax rate, in the presence of distortionary taxes is carried out. It is subject to maximization of household utility and requirement of government revenue by expanding on existing assumptions in a Ramsey setting. This analysis would help to understand the relationship between marginal social cost and Pigouvian tax in the presence of distortionary taxes when the household preferences are considered heterogenous as opposed to homogenous and marginal social cost considered as a variable, which depends on the Pigouvian tax, as opposed to being assumed as a constant.

Literature Review

The literature pertaining to Pigouvian tax dates back to the 1920's when economist Arthur Pigou first posed the idea in "The Economics of Welfare"

(Pigou, 1920). It is almost a century old idea posited by Pigou which has been either criticized or idealized and supported by various economists. There have been many contributions to the development of Pigou's idea of incorporating a tax to eliminate social costs.

According to Pigou (1920), the first-best tax on pollution is equal to the marginal social cost without the condition of no revenue requirement, or the use of lump-sum taxes by the government. In other words, Pigou proposed internalizing externalities via an optimum tax, called the Pigouvian tax. Such that the externality generating good or service production is decreased to the point that the marginal revenue equals the social marginal cost and thus, social welfare is maximized. The proposal of Pigou was a first–best remedy which, in the absence of distortionary taxes in the economy, moves the competitive equilibrium of the economy to its Pareto-efficient frontier. In a second–best environment, this is modified. Building upon the first best remedy, a significant contribution was made by Lipsey et al. (1956) on the theory of second best. The method of second best was used in the literature following this to take into consideration the presence of distortionary structures when calculating optimum solutions.

Sandmo (1975) considers indirect taxation to correct inefficiencies of resource allocation. The author, by analyzing second-best optimal tax structure, introduces the term of additive property. Among those who criticized Pigou's theory were Bovenberg, Mooij and Goulder. According to Bovenberg et al. (1994) and Bovenberg et al. (1996), the optimal environmental tax rate should be below the original Pigouvian tax rate (which fully internalizes the marginal social damage from pollution) in the presence of distortionary taxes in a second best setting. Fullerton et al. (1997) illustrates the concept of presumptive tax which, if it is not possible to tax commodities which create negative externalities, then complementary goods should be taxed. Cremer et al. (1998) analyzed the properties of optimal commodity and income taxes in the presence of externalities. Cremer et al. (2001) mentions about embodying both corrective and optimal tax objectives under a second-best tax rule.

The results obtained by Broadway et al. (2008) is of significant importance to the research we conducted since we built upon the models presented by these authors. The authors mathematically derive the relationship between Pigouvian tax and marginal social cost in the presence of other distortionary taxes such as commodity taxes and wage tax under given assumptions. These assumptions include homogeneity of households in the case of commodity taxes and marginal social cost being considered a constant in the presence of wage taxes. However, in our research we take it a step further by relaxing these assumptions. In the presence of commodity taxes, we consider heterogeneity of households as opposed to homogeneity. Also, in the presence of wage taxes the marginal social cost is considered as a variable that varies depending on the wage tax as opposed to being considered as a constant. It is pragmatic to consider heterogeneity in its simplest form by considering two groups since households in the complex world have heterogeneous preferences. Also, it is important to consider the dynamics of the marginal social cost when a Pigouvian tax is imposed. It is pragmatic to state that the marginal social cost would vary depending on the Pigouvian tax and not remain a constant as assumed in the literature.

Throughout the years, economists starting from Pigou (1920), to Coase (1960), to Sandmo (1975), to Stiglitz, (1987), to Bovenberg et al. (1994), to Cremer et al. (2001), to Boadway et al. (2008), and so many others, have made significant contributions in this regard to find the optimal tax structure that would incorporate the Pigouvian tax alongside other distortionary taxes such as commodity tax and wage tax subject to various conditions and restrictions in order to maximize utility.

The analysis behind relaxing the above mentioned two assumptions not being addressed in Boadway's paper led us to analyze the relationship between Pigouvian tax and Marginal Social Cost when the above two assumptions were relaxed. Therefore, more recent literature was not applicable in this case as an extension on Boadway (2008) paper was not carried out post 2008, and more recent literature focused upon other segments of Pigouvian taxes such as impact bonds, redistribution of income mobilized via Pigouvian tax, Pigouvian taxes internalizing social cost pertaining to platinum group element emissions, meat tax, managing credit booms via Pigouvian tax etc. and addressed similar relationships by adopting different models under different circumstances.

Research Methodology

The relationship between Pigouvian tax (t_p) and marginal social cost (β) was analyzed using two mathematical models in the presence of distortionary taxes, namely commodity and wage tax by relaxing the two assumptions made in the paper by Boadway et al. (2008). This paper tries to expand on the literature by addressing the following.

1. The first model looked into the relationship between t_p and β with heterogeneous household preferences, constant β, and in the presence of commodity tax.

2. The second model analyzed the relationship between t_p and β with varying β that depended on t_p such that $-1 < \beta'(t_p) < 0$, homogeneous household preferences, and in the presence of wage tax.

It was assumed that the utility function pertaining to households represent a quasilinear preference in leisure (l) (Varian, 1992). Each household's utility function is given by $U_C(C) + U_D(D) + E - L$(Broadway et al., 2008). Where, $U_C(C)$ and $U_D(D)$ represent increasing, strictly concave utilities of clean good C and dirty good D, respectively and $L = T - l$ where L represents labour supply and T represents total time available. Further, environmental quality $E = E_A - \beta ND$, where E_A is environmental quality in the absence of pollution, and N is the number of households (Broadway et al., 2008). By specifying the preferences as quasilinear in leisure, demand for goods depends only on its own prices relative to wage rate, and not on either income or prices of other goods. The only effect of a wage change on one's labor supply decision is the substitution effect (Mascollel et al., 1995). These assumptions help to obtain explicit solutions for the optimal Pigouvian tax rates on a polluting good.

The Lagrangian multiplier is a concept in mathematical optimization, which enables to find the local maximum/minimum of a function subject to equality constraints. Therefore, it was used to mathematically derive the relationship between t_p and β, subject to constraints such as shadow price of government revenue (λ_2) and utility maximization (marginal utility of income given by λ_1) for N in the presence of commodity taxes (t_C). λ_2 is important since it signifies the Ramsey component of the tax (Ramsey, 1927). A polluting good comprises of both Ramsey and Pigouvian effect. The Pareto efficient results obtained below are a result of theory of second best, since the relationship is obtained in the presence of distortionary taxes (Lipsey et al., 1956).

Pigouvian Tax in the Presence of Commodity Taxes

According to Broadway et al. (2008), the mathematical derivation provides us with the basic model for the relationship between Pigouvian tax and marginal social cost in the presence of commodity taxes with homogeneous preferences and constant marginal social cost as,

$$t_P = \frac{\varepsilon N \beta}{(\varepsilon + 1)\lambda_2 - \lambda_1}$$

Where ε is price elasticity of demand.

However, the above result has been derived considering only one group of households (i.e. homogeneous preferences), comprising of a total N number of households, with similar wage rates w.

In the society, this is not the case. Therefore, an analysis was carried out by relaxing this assumption.

Consider a two-household group model namely N_1 and N_2 with wage rates w_1 and w_2, with $w_2 > w_1$ in the presence of commodity tax. This would represent the heterogeneity of household preferences, which has not been considered in previous literature when analyzing the relationship between t_P and β in the presence of commodity taxes.

Let $C_i = (C_1, C_2)$ where C_1 represents the amount of clean goods C bought by household group N_1 and C_2 be the amount of clean goods C bought by household group N_2. In similar fashion $D_i = (D_1, D_2)$ vector can be defined as the amount of dirty goods D bought by each household groups N_1 and N_2. $U_C(C_i)$ would represent the utility of consuming good C_i ($i \in \{1,2\}$).

L_i represents the labor supply of L_1 or L_2 pertaining to household groups N_1 and N_2 respectively**.** The commodity taxes, t_C and t_D, remain the same for either household group. The Lagrangian function for households would then be,

$$L_1(C_i, D_i, L_i, \lambda_i) = U_C(C_i) + U_D(D_i) + E - L_i - \lambda_i((1 + t_C) \times C_i + (1 + t_D) \times D_i - w_i L_i) \quad (1)$$

By first order conditions, $C_i = C_i(\frac{1+t_C}{w_i})$, $D_i = D_i(\frac{1+t_D}{w_i})$ and $\lambda_i = \frac{1}{w_i}$ (2)

Therefore, indirect utility: $v_i\left(\frac{1+t_C}{w_i}, \frac{1+t_D}{w_i}\right) + E$, where $E = E_A - \beta(N_1 D_1 + N_2 D_2)$ (3)

By Envelope theorem, $\frac{\partial v_i}{\partial t_C} = -\frac{C_i}{w_i}$ and $\frac{\partial v_i}{\partial t_D} = -\frac{D_i}{w_i}$ (4)

Now using the Lagrangian function for the government,

$$L_2(t_C, t_D, \lambda_3) = \rho_1 N_1[v_1(.) + E_A - \beta(N_1D_1 + N_2D_2)] + \rho_2 N_2[v_2(.) + E_A - \beta(N_1D_1 + N_2D_2)] + \lambda_3[t_C(N_1C_1 + N_2C_2) + t_D(N_1D_1 + N_2D_2) - R] \quad (5)$$

Where, ρ_1 and ρ_2 are arbitrary social weights chosen such that the government can redistribute from high to low wage and λ_3 is shadow price of government revenue. By first order conditions, (using the fact that $\lambda_i = \frac{1}{w_i}$)

$$\frac{\partial L_2}{\partial t_C} = \rho_1 N_1 \left(\frac{-C_1}{w_1}\right) + \rho_2 N_2 \left(\frac{-C_2}{w_2}\right) + \lambda_3 \left[(N_1C_1 + N_2C_2) + t_C\left(\frac{N_1C_1'}{w_1} + \frac{N_2C_2'}{w_2}\right)\right] = 0 \quad (6)$$

Therefore,

$$\frac{\partial L_2}{\partial t_C} = -(\rho_1 N_1 C_1 \lambda_1 + \rho_2 N_2 C_2 \lambda_2) + \lambda_3\big((N_1C_1 + N_2C_2) + t_C(N_1C_1'\lambda_1 + N_2C_2'\lambda_2)\big) = 0 \quad (7)$$

By dividing above equation by $N_1C_1 + N_2C_2 \neq 0$;

$$\frac{-(\rho_1 N_1 C_1 \lambda_1 + \rho_2 N_2 C_2 \lambda_2)}{N_1C_1 + N_2C_2} + \lambda_3[1 + \frac{t_C\big(N_1C_1'\lambda_1 + N_2C_2'\lambda_2\big)}{N_1C_1 + N_2C_2} = 0 \quad (8)$$

Consider elasticity $\varepsilon_C = \frac{C_1'(.)}{C_1(.)}\frac{1+t_C}{w_1} = \frac{C_2'(.)}{C_2(.)}\frac{1+t_C}{w_2}$ (9)

Then notice that $\frac{C_1(.)\varepsilon_C}{w_1} = C_1'(.)\lambda_1$ and $\frac{C_2(.)\varepsilon_C}{w_2} = C_2'(.)\lambda_2$. Also let $\overline{\alpha_c} = \frac{(\rho_1 N_1 C_1 \lambda_1 + \rho_2 N_2 C_2 \lambda_2)}{N_1C_1 + N_2C_2}$

Therefore, $-\overline{\alpha_c} + \lambda_3\left[1 + \frac{t_C}{(1+t_C)}\varepsilon_C\right] = 0$ (10)

$$\frac{\partial L_2}{\partial t_D} = \rho_1 N_1\left(\frac{-D_1}{w_1}\right) + \rho_2 N_2\left(\frac{-D_2}{w_2}\right) - \beta(\rho_1 N_1 + \rho_2 N_2)\left(\frac{N_1D_1'}{w_1} + \frac{N_2D_2'}{w_2}\right) + \lambda_3[(N_1D_1 + N_2D_2) + t_D\left(\frac{N_1D_1'}{w_1} + \frac{N_2D_2'}{w_2}\right)] = 0 \quad (11)$$

Dividing this equation by $N_1D_1 + N_2D_2 \neq 0$

$$\frac{-(\rho_1 N_1 D_1 \lambda_1 + \rho_2 N_2 D_2 \lambda_2)}{N_1D_1 + N_2D_2} - \beta(\rho_1 N_1 + \rho_2 N_2)\left(\frac{N_1D_1'\lambda_1 + N_2D_2'\lambda_2}{N_1D_1 + N_2D_2}\right) + \lambda_3\left[1 + t_D\left(\frac{N_1D_1'\lambda_1 + N_2D_2'\lambda_2}{N_1D_1 + N_2D_2}\right)\right] = 0 \quad (12)$$

Let elasticity $\varepsilon_D = \frac{D_1'(.)}{D_1(.)}\frac{1+t_D}{w_1} = \frac{D_2'(.)}{D_2(.)}\frac{1+t_D}{w_2}$. Hence $\frac{D_1(.)\varepsilon_D}{w_1} = D_1'(.)\lambda_1$ and $\frac{D_2(.)\varepsilon_D}{w_2} = D_2'(.)\lambda_2$

Also let $\overline{\alpha_D} = \frac{(\rho_1 N_1 D_1 \lambda_1 + \rho_2 N_2 D_2 \lambda_2)}{N_1 C_1 + N_2 C_2}$ and $\overline{N} = \rho_1 N_1 + \rho_2 N_2$

Therefore, $-\overline{\alpha_D} + \beta\overline{N}\frac{\varepsilon_D}{1+t_D} + \lambda_3\left[1 + \frac{t_D}{1+t_D}\varepsilon_D\right] = 0$ (13)

Since preferences are homothetic in C and D and separable in leisure, it implies $\frac{C_1}{D_1} = \frac{C_2}{D_2}$ so that $\overline{\alpha_D} = \overline{\alpha_c} = \alpha$. Also assume that $\varepsilon_D = \varepsilon_C = \varepsilon$.

Thus, the above equations reduce to

$-\alpha + \lambda_3\left[1 + \frac{t_C}{(1+t_C)}\varepsilon\right] = 0$ (14)

$-\alpha + \beta\overline{N}\frac{\varepsilon}{1+t_D} + \lambda_3\left[1 + \frac{t_D}{1+t_D}\varepsilon\right] = 0$ (15)

From which the equations for tax on good C and D are obtained,

$t_C = (\frac{\alpha-\lambda_3}{\lambda_3})\frac{1+t_C}{\varepsilon}$ and $t_D = \left(\frac{\alpha-\lambda_3}{\lambda_3}\right)\frac{1+t_D}{\varepsilon} + \frac{\beta\overline{N}}{\lambda_3}$ (16)

Suppose $t_P = t_D - t_C$. Then the equation for the Pigouvian component of the tax is obtained in this case as,

$$t_P = \frac{\beta\overline{N}\varepsilon}{[(\varepsilon+1)\lambda_3 - \alpha]}$$

Pigouvian Tax in the Presence of Wage Taxes

According to Broadway et al. (2008) paper the relationship between Pigouvian tax and marginal social cost in the presence of wage taxes with constant marginal social cost and homogeneous preferences is given by,

$$t_P = \frac{N\beta\varepsilon(1-t_w)}{\lambda_2(\varepsilon+1) - (1-t_w)\lambda_1}$$

Now consider a deviation from the above model.

That is, consider the marginal social cost β as a function of t_p such that $-1 < \beta'(t_p) < 0$, which represents marginal social cost being considered as a variable that depends on the Pigouvian tax as opposed to being considered a constant. Which is something that has not been addressed. The initial equations that were obtained by using the Lagrangian equation for household will remain the same. Thus,

$$C = C(\frac{1}{(1-t_w)w}) \text{ , } D = D(\frac{1+t_P}{(1-t_w)w}) \text{ and } \lambda_1 = \frac{1}{(1-t_w)w} \quad (17)$$

Also, the results from the Envelope theorem would also not change.

$$\frac{\partial V}{\partial t_P} = -\lambda_1 D(.) = \frac{-D(.)}{(1-t_w)w} \text{ and } \frac{\partial V}{\partial t_w} = -\lambda_1 wL(.) = \frac{-L(.)}{(1-t_w)} \quad (18)$$

By use of the Lagrangian equation on the government objective function,

$$L_2(t_P, t_w, \lambda_2) = N\left[V(.) + E_A - \beta(t_p)ND(.)\right] + \lambda_2[NC\frac{t_w}{1-t_w} + ND\frac{t_P+t_w}{1-t_w} - R] \quad (19)$$

The above equation differs from the model where the marginal social cost was considered a constant. In this model it is considered as a function of the Pigouvian tax itself. By the first order conditions the following expression is derived,

$$\frac{\partial L_2}{\partial t_P} = \frac{-ND}{(1-t_w)w} - \frac{\beta N^2 D'}{(1-t_w)w} - \beta' N^2 D + \lambda_2\left[\frac{ND}{1-t_w} + N\frac{t_P+t_w}{(1-t_w)^2 w}D'\right] = 0 \quad (20)$$

Thus resulting in, $\frac{-D}{w} - \frac{\beta ND'}{w} - \beta' N(1-t_w)D + \lambda_2[D + \frac{t_P+t_w}{(1-t_w)w}] = 0$ (21)

Dividing this equation by D and using the equation for elasticity and marginal utility of income, the following equation is obtained,

$$-(1-t_w)\lambda_1 - \beta N\varepsilon\frac{(1-t_w)}{(1+t_P)} - \beta' N(1-t_w) + \lambda_2[1 + \frac{\varepsilon(t_P+t_w)}{(1+t_P)}] = 0 \quad (22)$$

$$\frac{\partial L_2}{\partial t_w} = N\left[\frac{-L}{(1-t_w)} - \frac{\beta ND'(1+t_P)}{w(1-t_w)^2}\right] + \lambda_2[NC\frac{(1-t_w)-(t_w)(-1)}{(1-t_w)^2} + \frac{Nt_w}{(1-t_w)}C'\frac{1}{(1-t_w)^2 w} + \frac{ND(1-t_w)(1)-(t_w+t_P)(-1)}{(1-t_w)} + \frac{N(t_w+t_P)D'(1+t_P)}{(1-t_w)w(1-t_w)^2} = 0 \quad (23)$$

And thus, by similar substitutions made the following is obtained,

$$\frac{-C}{w} + \lambda_2\left[C + \frac{C' t_w}{(1-t_w)w}\right] + (1+t_P)\left[\frac{-D}{w} - \frac{\beta N D'}{w} + \lambda_2 D + \lambda_2 \frac{D'(t_P+t_w)}{(1-t_w)w}\right] = 0 \quad (24)$$

Therefore, by substituting above expression, the following is obtained,

$$-(1-t_w)\lambda_1 + \lambda_2(1+\varepsilon t_w) + (1+t_p)\beta' N \frac{D(.)}{C(.)}(1-t_w) = 0 \qquad (25)$$

Now by multiplying equation (25) by $(1+t_P)$, and rearranging the terms,

$$t_P[-(1-t_w)\lambda_1 + \lambda_2(1+\varepsilon) - \beta' N(1+t_w)] = \lambda_1(1-t_w) - \lambda_2(1-\varepsilon t_w) + N\beta\varepsilon(1-t_w) + N\beta'(1-t_w) \qquad (26)$$

Therefore, the Pigouvian tax in this case would be,

$$t_P = \frac{\lambda_1(1-t_w) - \lambda_2(1+\varepsilon t_w) + N(1-t_w)[\beta\varepsilon + \beta']}{[\lambda_2(1+\varepsilon) - (1-t_w)\lambda_1 - \beta' N(1-t_w)]}$$

Results and Discussion

The results derived were mathematically proven above as it is a mathematical based analysis. The Lagrangian Multiplier approach was adopted as it is an optimization problem. Based on the mathematical derivations, the objective of the first model is to determine the optimum Pigouvian tax rate in the presence of commodity taxes when households are determined as heterogeneous. In both prior models, households were considered as homogeneous entities. This model is an aberration from that. This model considers heterogeneous households. It considers two groups of households namely, $N_1$and N_2. These two groups earn two different wage rates, namely w_1 and w_2. It is important to note that w_2>w_1. However, for calculation simplicity stake the β value is considered as constant.

$$t_P = \frac{\beta\overline{N}\varepsilon}{[(\varepsilon+1)\lambda_3 - \alpha]}$$

This equation is also analogous to the expression that was obtained by Broadway et al. (2008), which provided us with the basic model for the relationship between Pigouvian tax and marginal social cost in the presence of commodity taxes with homogeneous preferences and constant marginal social cost, except for the changes in λ_3, α and $\overline{N}$.

Note that at $\varepsilon = -1$, $t_P = \frac{\beta \bar{N}}{\alpha} = \frac{\beta}{\alpha}(\rho_1 N_1 + \rho_2 N_2) = \frac{\beta}{\alpha}\rho_1 N_1 + \frac{\beta}{\alpha}\rho_2 N_2$, which is different to that of Broadway et al. (2008) model at $\varepsilon = -1$, where $t_P = \frac{\beta N}{\lambda_1}$. In the model we derived for t_P in the presence of commodity taxes, when we consider unitary elasticity the Pigouvian tax depends on the marginal social weights assigned to each household group since β, N_1, N_2 are all constants and optimum value of α was pre-determined when considering the first order conditions in the Lagrangian function. Thus, the Pigouvian tax rate at unitary elasticity is an important observation.

Next it is important to analyze the sensitivity in the model we derived. That is, the sensitivity of the Pigouvian tax rate with regards to elasticity and the shadow price of government revenue. Therefore, by partially differentiating t_P, with respect to ε and λ_3,

$$\frac{\partial t_P}{\partial \varepsilon} = \frac{(\lambda_3 - \alpha)\beta \bar{N}}{[(\varepsilon+1)\lambda_3 - \alpha]^2} \text{ and } \frac{\partial t_P}{\partial \lambda_3} = \frac{-\beta \bar{N}(\varepsilon+1)\varepsilon}{[(\varepsilon+1)\lambda_3 - \alpha]^2}$$

Since α and $\bar{N}$ depend on the social weights assigned to them, the sensitivities also depend indirectly on the weights been assigned as well.

Note that, as before in this case also when unitary elasticity is considered the sensitivity of the Pigouvian tax to the shadow price of government revenue is zero. In the same case the sensitivity of Pigouvian tax rate to elasticity $\frac{\partial t_P}{\partial \varepsilon} = \frac{(\lambda_3 - \alpha)\beta \bar{N}}{\alpha^2} > 0$ assuming that $\lambda_3 > \alpha$. Also, assuming that $\alpha < \lambda_1$(marginal utility in this model is less than the marginal utility of the first model) then the sensitivity is higher in this model compared to the model obtained by Broadway et al. (2008) and vice versa.

$\frac{\partial t_P}{\partial \lambda_3} = \frac{-\beta \bar{N}(\varepsilon+1)\varepsilon}{[(\varepsilon+1)\lambda_3 - \alpha]^2}$ value depends on the elasticity. If $-1 < \varepsilon < 0$, then it is positive. However, if elasticity is greater than -1, then the value is negative. This value compared to the value obtained in Broadway's model, depending on $\bar{N}$, α and λ_3, will help determine which Pigouvian tax rate is more sensitive to the shadow price of government revenue under each scenario laid out.

Subtle changes were made to the model obtained by Broadway et al. (2008) where the model we derived looked at heterogenous households as opposed to homogeneous households. The changes were then compared with each other to get an understanding and an insight into Pigouvian tax rates in the presence of commodity taxes. Further research could be carried out where in the model we derived the β could be considered as a function of tax on good D rather than a constant. Such alterations enable other researchers to further delve into the theory of Optimal Pigouvian taxes in the presence of commodity taxes given that it is analyzed in a Ramsey environment.

The objective of the second model was to understand the behavior and nature of the Pigouvian tax when the marginal social cost is considered as a function of the Pigouvian tax in the presence of a wage tax t_w. This is a somewhat complex model. However, the calculation was made simple, and the derivation was not difficult.

The Pigouvian tax obtained in this scenario was,

$$t_P = \frac{\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w)+N(1-t_w)[\beta\varepsilon+\beta']}{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_w)]} \quad (27)$$

The numerator can also be re-written in the following manner, by substituting

$(1+t_p)\beta' N\frac{D(.)}{C(.)}(1-t_w)$ for the expression $\lambda_1(1-t_w)-\lambda_2(1-\varepsilon t_w)$,

$$t_P = \frac{(1+t_p)\beta' N\frac{D(.)}{C(.)}(1-t_w)+N(1-t_w)[\beta\varepsilon+\beta']}{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_w)]} \quad (28)$$

Unlike the other equations, the Pigouvian tax component in this case cannot be separated easily since t_P determines the demand for goods C and D. Also, it determines β. Since all three of these variables depend on the Pigouvian tax t_P, it is not possible in this case to separate them. However, a basic understanding can be obtained from the expression derived for the Pigouvian tax component in this scenario.

From the model obtained by Broadway it is possible to obtain an equation similar to the equation above. It would be in the following form,

$$t_P = \frac{\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w)+N\beta\varepsilon(1-t_w)}{\lambda_2(\varepsilon+1)-(1-t_w)\lambda_1} \quad (29)$$

which is very much similar to equation (28). However, why the equation that was obtained in model by Broadway et al. (2008) is different to that of equation (29) above is that, $\lambda_1(1-t_w)-\lambda_2(1-\varepsilon t_w)$ was proven to equal zero. However, in model we derived, the same equation does not equal to zero. It equals to $(1+t_p)\beta' N \frac{D(.)}{C(.)}(1-t_w)$. Which implies that the Pigouvian tax depends on the optimum demand for goods D and C. Where D in turn depends on the Pigouvian tax. However, the optimum demands were calculated when the Lagrangian equation was used for the objective function of households in order to determine the optimum demands. If $(1+t_p)\beta' N \frac{D(.)}{C(.)}(1-t_w)$ were to be substituted instead of $\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w)$ then t_P would be,

$$t_P = \frac{\beta' N(1-t_w)(\frac{D}{C}+1) + N\beta\varepsilon(1-t_w)}{\lambda_2(\varepsilon+1)-(1-t_w)\lambda_1-\beta' N(1-t_w)(\frac{D}{C}+1)}$$

Keeping in mind that $-1 < \beta' < 0$.

Clearly the Pigouvian tax in this case is smaller in value compared to the Pigouvian tax obtained in model obtained by Broadway et al. (2008). Why? Since the term $\beta' N(1-t_w)(\frac{D}{C}+1)$ is added to the denominator due to the fact that $-1 < \beta' < 0$. Also, the same term is deducted from the numerator.

Thus, when the marginal damage to the environment (β) is a constant then the Pigouvian tax is larger compared to when the marginal damage to the environment is considered as a function of t_P., which is an important observation. β may also be considered as a function of both t_P and t_w. Since the tax on wages and tax on consuming dirty goods would eventually have an impact on curtailing the usage of goods harming the environment due to the reduction in disposable income with respect to tax on wage.

In such a case suppose $\beta = \beta(\frac{t_P}{1-t_w})$ such that $-1 < \frac{\partial\beta}{\partial t_P} < 0$ and $-1 < \frac{\partial\beta}{\partial t_w} < 0$. That is marginal damage to the environment decreases as tax on wage and tax on dirty good increases. The same calculation can be observed with minor alterations. The important expression that is obtained in this calculation is the following,

$$\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w) = [-\beta' N t_P \frac{D}{C} + \frac{(1+t_P)\beta N D'}{cw}]$$

By substituting this expression in $t_P = \frac{\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w)+N[\beta\varepsilon(1-t_w)+\beta']}{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N]}$ the following equation is obtained for Pigouvian tax in the scenario where the marginal damage to the environment is not only depending on tax on the dirty good but also the wage tax. After a few mathematical manipulations the following important expression is obtained.

$$t_P = \frac{N\beta\varepsilon(1-t_w)(\frac{D}{C}+1)+\beta' N}{\left[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_P\frac{D}{C})\right]}$$

It is important to compare this expression, which is the Pigouvian tax component when the marginal damage is a function of both t_P and t_w, with the Pigouvian tax obtained when the marginal damage is a function of t_P only. This provides a better understanding of the behavior of Pigouvian tax in differing environments, even when changes as subtle as this are made. It is not very clear as to which tax is larger since it depends on the values of tax on wage and the values of the demands of goods D and C.

Next it is important to analyze the sensitivity in the model we derived. That is, the sensitivity of the Pigouvian tax rate with regards to elasticity and the shadow price of government revenue. Therefore, by partially differentiating t_P, with respect to ε and λ_2,

$$\frac{\partial t_P}{\partial \varepsilon} = \frac{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_w)](N\beta(1-t_w)-\lambda_2 t_w)-\lambda_2[N\beta\varepsilon(1-t_w)+[\lambda_1(1-t_w)-\lambda_2(1+\varepsilon t_w)]+N\beta'(1-t_w)]}{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_w)]^2} \tag{30}$$

$$\frac{\partial t_P}{\partial \lambda_2} = \frac{-N\beta\varepsilon(1-t_w)(1+\varepsilon)-\lambda_1\varepsilon(1-t_w)^2-N\beta'\varepsilon(1-t_w)^2}{[\lambda_2(1+\varepsilon)-(1-t_w)\lambda_1-\beta' N(1-t_w)]^2} \tag{31}$$

Implications and Conclusions

The overall objective of this research was to scrutinize the changing dynamics of Pigouvian tax rate in relation to marginal social cost within the presence of an already existing distortionary tax system in a Ramsey environment. The distortionary tax system was not taken up as a whole i.e., all the distortionary taxes were not considered together in a single system, but rather it was segregated into commodity taxes, and wage taxes such that the tax structure had only one distortionary tax at a time alongside the Pigouvian tax. This was done in order to obtain a better understanding of the optimal Pigouvian tax behavior in the presence of individual distortionary tax systems in place.

In the model (presented by Boadway) for Pigouvian tax in the presence of commodity taxes, the assumption on homogeneity of households was relaxed. Practically in an economic sense, it is not possible to come across a homogeneous set of households. It is safe to assume that almost all the households are heterogenous. Therefore, this model focused upon finding the Pigouvian tax rate and its relationship to marginal social costs when the households are heterogeneous, while keeping the marginal damage to the environment constant. It was interesting to see that the Pigouvian tax rate was analogous to that of the model obtained by Broadway although the terms are different. If the elasticity is a negative one in both models, and the marginal utility of income in the model we derived is less than that of the model obtained by Broadway, then the value of the Pigouvian tax is higher in the model we derived compared to the model obtained by Broadway. This again is an important observation, since it tells the policy maker to be aware of the relevant details pertaining to not only the elasticity of the good but also have a certain understanding of the marginal utility of income of families when imposing such a tax.

The deviation from the model obtained by Broadway for Pigouvian tax in the presence of wage taxes is presented in the model we derived where the marginal damage to the environment is considered as a function of tax on dirty good. It was considered that when the tax rises the marginal damage reduces. One important observation is that in the model we derived the Pigouvian tax is greater in value than that of the optimal Pigouvian tax in the model obtained by Broadway under certain conditions. That is, the optimal Pigouvian tax in the presence of wage tax is higher when the marginal damage to the environment was considered a variable as opposed to a constant. However, it was observed that a comparison could not be drawn in the other occasion, for which raw data was required. The sensitivity of the Pigouvian tax to elasticity was lower in our model compared to the model obtained by Broadway in one case. That is the Pigouvian tax was more sensitive to elasticity in the presence of wage tax when the marginal damage was considered a constant as opposed to being considered as a variable. The other case required raw data in order to draw comparisons. Analyzing equation (31) showed us that the sensitivity of the Pigouvian tax to shadow price of government revenue, in absolute terms, was lower in our model compared to Broadway's model. That is to say, that when the marginal damage to the environment was considered a constant, the sensitivity of the Pigouvian tax was higher as opposed to when the marginal damage was considered to be a variable in the presence of wage taxes.

In the case where optimal Pigouvian tax is calculated in the presence of commodity taxes, policy makers should consider the following. In the case of inelastic demands, the Pigouvian tax would diverge positively from marginal social cost as government revenue requirements increase. The Pigouvian tax would diverge negatively from marginal social cost as revenue requirements increase in the case of elastic demands. In an example, when deciding upon introducing a carbon tax into a tax structure which already comprises of a commodity tax on fuel (which could be treated as an inelastic good), as the government revenue requirements increase, the carbon tax should be imposed at a higher level than the marginal social cost caused by the consumption of fuel in the economy. Also policy makers should consider the fact that if for example, the marginal damage caused by the release of toxic waste by a certain factory to the waterway reduced along with the increase of the tax imposed on the company factory, then what the research outcome posits is that the government should impose a relatively lower Pigouvian tax compared to the case where a Pigouvian tax is imposed when the marginal damage caused by the release of toxic waste to the waterway remained constant (that is to say the company does not reduce the amount of toxins released).

In the case of Pigouvian tax implemented in the presence of wage taxes, policy makers have to take into consideration the elasticity of the good. If the elasticity of the good is such that a change in price will not affect total revenue (i.e., unrelated goods), then the Pigouvian tax should be implemented equivalent to the marginal social cost. However, if these goods are either complementary or substitutable, then the Pigouvian tax should be set at a lower value compared to Pigouvian tax set under commodity tax structure. However, the sensitivity of the Pigouvian tax to elasticity was higher in the presence of commodity taxes as opposed to linear progressive income tax. Policy makers should also collect data pertaining to income tax, marginal social damages, shadow price of government revenue etc. in order to draw comparison between the optimal Pigouvian tax in the presence of commodity tax and in the presence of linear progressive income tax. This would enable them to come up with an optimal value for the Pigouvian tax.

In conclusion it is of vital importance to have policy dialogue pertaining to Pigouvian tax, since understanding the optimal value at which it should be imposed would help countries to raise more revenue and direct this revenue as cash transfers to those from rural society, which is another area of research where much research is being carried out. In the long run, having an understanding of an optimum Pigouvian tax (in the presence of distortionary taxes) would help to implement taxes such as carbon taxes in order to bring down emissions quickly and lower the cost of transition (moving to a low carbon economy as is the main priority to tackle climate change).

Further research could include a model that could be developed such, that it includes all three of these distortionary taxes along with the Pigouvian tax, thereby determining the optimal Pigouvian tax in the midst of three types of distortionary taxes. Econometric/Machine Learning based modeling based on data to analyze the relationship between Pigouvian taxes and Marginal social cost in situations where the mathematical models do not provide answers is also another aspect that could be further looked into in order to either disprove or further existing mathematical models. Also, the administrative cost pertaining to implementing Pigouvian taxes in this scenario can be considered in further research to determine the dynamics of Pigouvian taxes and how its relationship with marginal social cost would differ.

Declaration of Conflicting Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Authors' Contributions

TSV performed the mathematical analysis and was responsible for partially drafting the manuscript, K participated in its design and coordination and helped to draft the manuscript. All authors have read and approved the final version of the manuscript and agree with the order of presentation of the authors.

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