The Mortality Kinks Kinematics Involving Gauss Forward Interpolation

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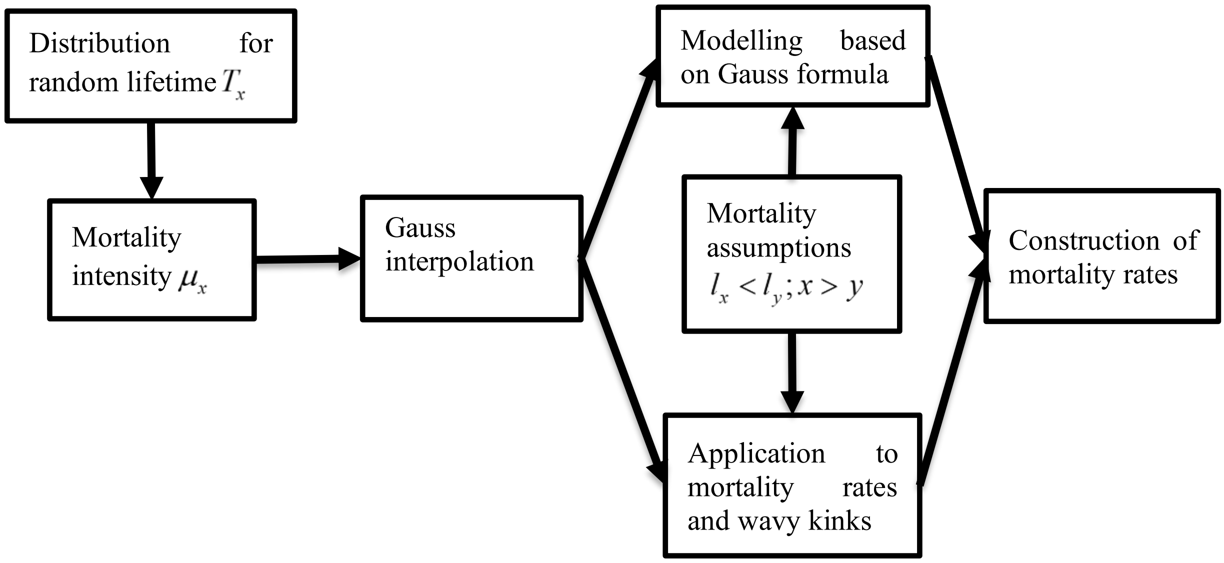

Abstract

The estimation of continuous death probability has motivated competitive interest in life assurance practice and mortality literatures with the study's specific goal of estimating the force of mortality with an acceptable level of accuracy. Using the non-parametric interpolation to model human mortality rates has several implications especially in terms of its effectiveness and practical applications in actuarial science, insurance and pension planning. In the context of mortality modeling, the model involves fitting mortality data to interpolation model that captures the forward-looking behavior of mortality rates over time. This model is capable of incorporating the future evolution of mortality rates based on past trends and economic conditions. Initiating Gauss forward interpolation process involves a crucial stage of theoretical work that yields much more valuable outcomes with tremendous practical significance to investigation of mortality behaviour. The objectives of the research are to detect kinks on instantaneous death rate and construct the corresponding mortality rates. From the results obtained, the modal age at death is around 84 for normal humans. Although the methodology has the potential to detect kinks due to random fluctuations or shocks in mortality rates that are difficult to predict deterministically otherwise, computational evidence confirm that the interpolative approach provides more accurate, improved, robust and reasonable lower mortality estimates compared to parametric results in Neil's. This implies that while mortality rates may follow broad trends, there can be unexpected changes due to external factors like pandemics, economic shifts or healthcare advancements.

Keywords: *Probability, Estimation, Life assurance, Mortality, Lifetime*

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1. Introduction

Wavy kinks refer to the phenomenon where mortality rates typically decreasing or steady suddenly kink at specific age points only to decrease or level off again. These kinks can occur at various ages but they are often observed at older ages such as around $80-120$ years. While the exact causes of wavy kinks are still being debated, plausible explanations are: (i) biases in mortality data collection or processing may contribute to the appearance of wavy kinks, (ii) differences in mortality rates between birth cohorts may lead to kinks in mortality rates at specific ages, (iii) the survival of certain subgroups within a population may influence mortality rates resulting in wavy kinks, (iv) the improvements in medical care and technology may lead to changes in mortality rates causing kinks at specific ages, (v) wavy kinks can also impact pension plan valuation as they can lead to changes in the expected number of deaths and consequently, the expected pension payments, while this can result in significant actuarial losses affecting the overall financial health of the pension plan, (vi) actuaries and life insurers must comply with regulatory requirements, such as Solvency II and IFRS 17 which demand accurate mortality modelling and risk assessment. Failing to account for wavy kinks can lead to non-compliance and potential penalties. Understanding and modelling wavy kinks can help improve mortality forecasting, enabling actuaries to better anticipate future mortality trends and make more informed decisions. In order to detect the presence of kinks, interpolation is deployed in mortality rate modelling.

Mortality modelling is a crucial aspect of actuarial science as it enables the estimation of future mortality rates and the calculation of life insurance reserves, annuity values, and pension fund liabilities. Traditional mortality models such as the parsimonious models have been widely used in the industry. However, these models often suffer from limitations, such as assuming a constant mortality rate improvement over time. Gauss interpolation, a specialized technique has been proposed as an alternative approach to mortality modelling.

Interpolation refers to the process of estimating a value between two known values in a sequence (Ndu, Nwuju & Bunonyo, 2019). According to literature, (DeBoor, 1978; Das and Chakrabarty 2016), in the context of data points, it involves obtaining an intermediate value of an actuarial function based on a set of given values. As observed by Ndu et al., (Ndu, Nwuju, and Bunonyo 2019), when there is a gap in survival data and data is available on both sides of the gap, Gauss interpolation can be used to estimate the corresponding mortality rates for the gap. A model was developed (Hudec 2017) for the force of mortality μ_x in (1)

$$\mu_x = a(x) + \sum_{j=1}^{d} b_j(x)x^j \tag{1}$$

using local polynomial methods where $a(x)$ and $b_j(x)$ are regression coefficients for each target age x. A major problem here is that the model cannot be used to compute both life annuity and life insurance functions but only limited to mortality intensities.

However, an experimented alternative application of survival function l_x in life insurance using a quadratic model of the form $l_x = ax^2 + bx + c$ through which the mortality intensity function μ_x was modelled (Pavlov and Mihova, 2018). A demerit in the work is that the quadratic model is not fit for continuous life insurance functions; moreover, quadratic functions are not capable of addressing the permissible age of validity in life insurance.

The upper and lower bounds for the continuous life annuity $\bar{a}_x$ and life insurance $\bar{A}_x$ functions to address issues relating to incomplete life insurance data was derived (Souza, 2019). The bounds were not tested under any known parsimonious mortality laws. Furthermore, the method of the integrated hazard function through any known parametric or polynomial mortality intensities was not considered before the calculation of life annuity and life insurance benefits.

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A crucial issue in life contingencies borders on the estimation of nonlinear mortality intensity functions which are analytically derived to develop products for life offices. In two accounts of work, (Dickson et al., 2013; Siswono et al., 2021), it was concluded that the linear interpolation function.

$$l_{x+\xi} = (1-\xi)l_x + \xi l_{x+1} \tag{2}$$

was deployed to address basic computational problems to ease out complex analytical estimations. The authors used the linear interpolation to simplify nonlinear mortality functions in mortality computations but unknown to the authors, the survival function $l_x \in C^K$ that is l_x is continuously differentiable up to order K. The implication is that the differential coefficient of the linear interpolation function becomes zero as from the second derivative and consequently higher order Taylor's series expansion of mortality function is intractable.

Mortality refers to the level of death within a population, measured by the number of deaths and the death rates characteristic of that population during a specific period (Hossain et al 2023). Others (Congdon 1993; Rabbi and Karmaker, 2013) describe mortality rate as the number of deaths per unit of time in a population, scaled to the size of that population. From their perspective, a life table representing a population's mortality provides insights into its survivorship and mortality experiences. As noted by Hsieh (Hsieh, 1991), the information in a life table includes instantaneous mortality rates, probabilities of dying and surviving, and expected years of life remaining. Actuaries primarily use life tables to calculate annual provisions for life insurance and estimate the future value of retirement funds. Life tables are also crucial for determining the monthly premiums and contributions charged to policyholders.

The force of mortality refers to the instantaneous rate of death at a specific moment, given survival up until that point. As defined by researchers (Kostaki and Panousis, 2001; Kostaki and Panousis, 2019) the force of mortality in a decrement life table is a function of the number of survivors in a particular age group. Unlike interval measures, it is defined as a function of survivors at each specific age. A life table is a mathematical tool used in demography to track mortality or fertility trends for a cohort or the entire population. However, estimating the force of mortality is challenging unless there are methods for creating analytical models based on the number of survivors l_x at different ages.

Others (McCutcheon, 1983; Rabbi and Karmaker, 2013) foresaw this difficulty and proposed an interpolation technique to create mortality matrices, which was used to derive models for the instantaneous mortality rate. Several parametric functions have been developed in actuarial literature to model mortality rates. These models often involve more than two parameters, making parameter estimation challenging due to the lack of closed-form solutions. While maximum likelihood estimation is commonly used, it lacks the potential to solve parsimoniously system of first-order partial differential equations involving the estimation of the parametric aging parameter.

According to researchers (Putra et al., 2019), using maximum likelihood algorithms requires inputting initial parameter values and if these values are far from the true values, the results may be misleading, with the algorithms failing to converge. An alternative approach focuses on using cohort experience to set assumptions for mortality at advanced ages, rather than directly modeling old-age mortality for annuitants (Astuti et al., 2013). This method involves interpolating mortality experience from the central age range to the population mortality experienced at older ages, ensuring that the mortality rate trajectories for annuity holders align with those of the general population. This approach helps guide the choice of which population mortality table an annuity holder should use. Although mortality tables typically account for assumptions only up to a specified ultimate age, death rate probabilities at this age are often fixed, even if the developed model predicts a lower rate. Consequently, life expectancy curves can be estimated from these mortality models.

Actuaries have observed that mortality rates can be highly sensitive to changes in demographic and socioeconomic conditions. This observation stems from the principle of age-dependent mortality models, which have significantly impacted mortality forecasting. Parametric mortality models enable actuaries to assess the risks associated with uncertainties in mortality projections. However, intractable

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issues often arise in computing rates, particularly when estimating the force of death, which is difficult when data is limited or the intensity is hard to quantify.

The model developed by Lee and Carter has become one of the most widely used approaches for mortality forecasting (Lee and Carter, 1992). Although the model does not rely on polynomials, polynomial terms are often included to improve the model's fit and capture long-term mortality trends more effectively. Extensions of the Lee-Carter model typically involve adding polynomial adjustments to the parameters or incorporating polynomial terms to better account for mortality improvement trends, particularly when dealing with high levels of mortality estimation and survival function values. The survival function l_x is graduated at various ages and the underlying mathematical equations are not provided. Gauss interpolation is a method for interpolating data using a Gaussian function. It is particularly useful when the data is sparse or irregularly spaced. In the context of mortality modelling, interpolation can be used to estimate mortality rates at intermediate ages or time points, based on a set of observed mortality rates. An introduction to numerical interpolation and its application to mortality analysis has been provided (Benjamin and Pollard, 1980).

In one of the earliest applications of interpolation in mortality modelling (Elandt-Johnson and Johnson, 1980), the authors introduced interpolation to obtain mortality rates at intermediate ages, based on a set of observed mortality rates and demonstrated its potential in mortality modelling. The authors showed that Lagrange's interpolation can provide more accurate estimates of mortality rates, particularly at intermediate ages, compared to traditional methods. Several mortality models have been proposed that incorporate interpolation. For example, an analytical mortality model that uses polynomial interpolation to estimate mortality rates at intermediate ages has been developed (McCutcheon, 1983) where the author demonstrated that the model can provide more accurate estimates of mortality rates, particularly for populations with limited data. Similarly, an analytic mortality model was proposed (Elandt-Johnson and Johnson, 1980) that involves a six-point Lagrange's interpolation, demonstrating that it can provide more robust estimates of mortality rates compared to traditional models.

The estimation of the death rate intensity μ_x at any given time is a challenge that frequently arises in life and other circumstances. It was observed that unless l_x can be functionally expressed as a convergent series polynomial function, it becomes impossible to estimate the value of μ_x analytically from the first order ordinary differential equation defined by

$$\mu_x l_x = -\frac{dl_x}{dx} \text{ (Neil, 1977).}$$

This is specific because l_x defines the number of lives expected to survive to age x while μ_x is the instantaneous death rate. In this study (Neil, 1977), the author's goal was to effectively estimate μ_x at a using Taylor's series expansion based on the foundation of known mortality data and under the supposition that l_x is a convergent series polynomial function by interpolating at the commencement of the mortality table. However, if the mortality table is not based on a functional expression, it is impossible to determine the survival function l_x at a fractional age when it is required unless through linear interpolation. As a result, the objective is to use methods of interpolation to derive approximate values rather than their actual analytical values.

Modeling mortality rates using forward interpolation can be justified based on the following reasoning: Mortality rates presented in life tables exhibit smooth changes over time across age groups. Forward interpolation is particularly useful for smoothing mortality data, which is ideal when dealing with mortality rates that are continuous and smooth in nature. Interpolating method ensures that the estimated mortality rates are smoothly varying reflecting realistic changes over time across ages.

Mortality data may be incomplete or have gaps especially at specific ages. Forward interpolation can be deployed to estimate the missing data points. This is particularly useful in actuarial work where a full mortality table is often needed but not all ages or periods may have observed data.

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By interpolating using a reliable forward method, such gaps can be filled with reasonable estimates reducing bias in subsequent calculations.

The forward interpolation method specifically utilizes known data points from a sequence and then estimates future points using a weighted average of known data. This makes it especially effective for mortality data when there is a natural progression of mortality rates with age over time. Forward interpolation assumes that past values have an impact on future values which aligns with the nature of mortality trends.

The forward interpolation works by using a series of weights to adjust the estimated value. The weights are computed based on the differences between consecutive data points, ensuring that the interpolation is accurate and reflects the changes in the mortality rates as closely as possible. This method can provide a better fit for the observed mortality data particularly when mortality rates show non-linear trends. Forward interpolation is rooted in polynomial approximation which is robust for capturing trends in numerical data that vary smoothly.

Given that mortality rates often follow smooth and continuous functions (such as increasing with age), the technique can model these trends accurately and efficiently without requiring complex non-linear models. In actuarial science, the modeling of mortality rates is essential for determining life expectancies, insurance premiums and pension liabilities. Forward interpolation provides a relatively effective tool for estimating mortality rates when there is incomplete data or when smoothing is necessary. It helps in producing estimates for forward-looking projections allowing for better decision-making in financial and demographic planning.

If the available mortality data is limited, forward interpolation can provide good estimates by leveraging existing data points efficiently. The method does not require huge amounts of data making it suitable for situations where only limited historical mortality data is available. Forward interpolation is relatively straightforward to implement computationally. This is advantageous when dealing with large datasets or when speed and computational efficiency are important. Actuarial and demographic models often need to be implemented in environments where computational resources or time are limited.

In order to avoid the time-consuming parameter estimation, a viable option to model death rate and life expectancy curve is nonparametric forward estimation since it is also more widely applicable than parametric mortality models. The use of forward interpolation to model mortality rates is justified due to its ability to smoothly estimate missing values, handle incomplete or sparse data and provide reasonable approximations which align with the known trend of mortality data. It offers a balanced approach between accuracy, computational efficiency and simplicity, making it a valuable tool for modeling mortality rates in actuarial applications.

Theorem:

Let l be the survival function of age x, then

$$\Delta^r l_{x-1} = \Delta^r E^{-r} l_x \tag{3}$$

Proof:

$$\Delta^r l_{x-1} = l_x - r l_{x-1} + \frac{r(r-1)}{2} l_{x-2} + \ldots + (-1)^r l_{x-r} \tag{4}$$

The RHS is given as

$$\begin{aligned} & l_x - r l_{x-1} + \frac{r(r-1)}{2} l_{x-2} + \ldots + (-1)^r l_{x-r} \\ & = l_x - r E^{-1} l_x + \frac{r(r-1)}{2} E^{-2} l_x + \ldots + (-1)^r E^{-r} l_x \end{aligned} \tag{5}$$

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$$\Delta^r l_{x-1} = \left[1 - rE^{-1} + \frac{r(r-1)}{2}E^{-2} + ... + (-1)^r E^{-r}\right] l_x \quad (6)$$

$$\Delta^r l_{x-1} = \left(1 - E^{-1}\right)^r l_x \quad (7)$$

$$\Delta^r l_{x-1} = \left(1 - \frac{1}{E}\right)^r l_x = \left(\frac{E-1}{E}\right)^r l_x = \frac{\Delta^r}{E^r} l_x \quad (8)$$

$$\Delta^r l_{x-1} = \Delta^r E^{-r} l_x \quad (9)$$

QED

Theorem: Let l be the survival function of age x and l_0 be the radix, then

$$e^x \left(l_0 + x\Delta l_0 + \frac{x^2}{2!}\Delta^2 l_0 + ... \right) = l_0 + l_1 x + l_2 \frac{x^2}{2!} + ... \quad (10)$$

Proof:

The LHS is given as,

$$e^x \left(l_0 + x\Delta l_0 + \frac{x^2}{2!}\Delta^2 l_0 + ... \right) = e^x \left(1 + x\Delta + \frac{x^2}{2!}\Delta^2 + ... \right) l_0 \quad (11)$$

$$\Rightarrow e^x \left(l_0 + x\Delta l_0 + \frac{x^2}{2!}\Delta^2 l_0 + ... \right) = e^x e^{x\Delta} l_0 = e^{x(1+\Delta)} l_0 = \left(e^{xE}\right) l_0 \quad (12)$$

$$e^x \left(l_0 + x\Delta l_0 + \frac{x^2}{2!}\Delta^2 l_0 + ... \right) = \left[1 + xE + \frac{x^2 E^2}{2!} + ... \right] l_o \quad (13)$$

$$e^x \left(l_0 + x\Delta l_0 + \frac{x^2}{2!}\Delta^2 l_0 + ... \right) = l_0 + l_1 x + l_2 \frac{x^2}{2!} + ... \quad (14)$$

QED

2. Methodology

2.1 Gauss forward Interpolation

In order to derive close form mortality rate intensity formula, the following successive differencing is required.

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1} \quad (15a)$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad (15b)$$

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$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \tag{15c}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \tag{15d}$$

$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2} \tag{15e}$$

$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2} \tag{15f}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \tag{15g}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \tag{15h}$$

Use (5d) and (15h) to get the Newton's formula

$$\begin{aligned} y_p &= y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) \\ &+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_0) + \ldots \end{aligned} \tag{15i}$$

Substitute $\Delta^2 y_0$, $\Delta^3 y_0$ and $\Delta^4 y_0$ in (15i) to obtain

$$\begin{aligned} y_p &= y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) \\ &+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots \end{aligned} \tag{15j}$$

$$\begin{aligned} y_p &= y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}(\Delta^3 y_{-1}) \\ &+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + \ldots \end{aligned} \tag{15k}$$

Substitute $\Delta^4 y_{-1}$ to get

$$\begin{aligned} y_p &= y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}(\Delta^3 y_{-1}) \\ &+ \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-2}) + \ldots \end{aligned} \tag{15l}$$

$$y_p = \Delta^0 y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta y_{-2} + \ldots \tag{15m}$$

$$G_1 = p \tag{16}$$

$$G_2 = \frac{p(p-1)}{2!} = \frac{p^2 - p}{2!} \tag{17}$$

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$$G_3 = \frac{(p+1)p(p-1)}{3!} = \frac{p(p^2-1)}{3!} = \frac{(p^3-p)}{3!} \tag{18}$$

$$G_4 = \frac{(p+1)p(p-1)(p-2)}{4!} = \frac{(p^4-p^2-2p^3+2p)}{4!} \tag{19}$$

$$G_5 = \frac{p(p^2-1(p^2-4))}{5!} = \frac{(p^3-1)p(p^2-4)}{5!} = \frac{p^5-p^3-4p^3+4p}{5!} = \frac{p^5-5p^3+4p}{5!} \tag{20}$$

$$y_p = \frac{\Delta^0 y_0}{\int_1^\infty \frac{(\ln x)^0}{x^2}dx} + \frac{p}{\int_1^\infty \frac{(\ln x)^1}{x^2}dx}\Delta^1 y_0 + \frac{p(p-1)}{\int_1^\infty \frac{(\ln x)^2}{x^2}dx}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{\int_1^\infty \frac{(\ln x)^3}{x^2}dx}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{\int_1^\infty \frac{(\ln x)^4}{x^2}dx}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{\int_1^\infty \frac{(\ln x)^5}{x^2}dx}\Delta^5 l_{-3} \tag{21}$$

$$l_{0+p} = \frac{\Delta^0 l_x}{0!} + \frac{p\Delta^1 l_x}{1!} + \frac{p(p-1)\Delta^2 l_{x-1}}{2!} + \frac{(p+1)p(p-1)\Delta^3 l_{x-1}}{3!} + \frac{(p+1)p(p-1)(p-2)\Delta^4 l_{x-2}}{4!} + \frac{(p+2)(p+1)p(p-1)(p-2)\Delta^5 l_{x-2}}{5!} + \ldots + \frac{(p+1)p(p-1)(p-2)(p-3)\ldots(p-n+1)\Delta^n l_x}{n!} \tag{22}$$

Setting $p=t$ and $y=l$ in (22)

$$l_{0+t} = \frac{\Delta^0 l_x}{1!} + \frac{t\Delta^1 l_x}{1!} + \frac{t(t-1)\Delta^2 l_{x-1}}{2!} + \frac{(t-1)t(t+1)\Delta^3 l_{x-1}}{3!} + \frac{(t+1)t(t-1)(t-2)\Delta^4 l_{x-2}}{4!} + \frac{(t+2)(t+1)t(t-1)(t-2)\Delta^5 l_{x-2}}{5!} \tag{23}$$

Now $x+0+t = x+t$

$$l_{x+t} = \frac{\Delta^0 l_x}{1!} + \frac{t\Delta^1 l_x}{1!} + \frac{t(t-1)\Delta^2 l_{x-1}}{2!} + \frac{(t+1)t(t-1)\Delta^3 l_{x-1}}{3!} + \frac{(t+1)t(t-1)(t-2)\Delta^4 l_{x-2}}{4!} + \frac{(t+2)(t+1)t(t-1)(t-2)\Delta^5 l_{x-2}}{5!} \tag{24}$$

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$$l_{x+t} = \frac{\Delta^0 l_x}{0!} + \frac{t\Delta^1 l_x}{1!} + \frac{(t^2 - t)\Delta^2 l_{x-1}}{2!} + \frac{(t^3 - t)\Delta^3 l_{x-1}}{3!} + \frac{(t^4 - t^2 - 2t^3 + 2t)\Delta^4 l_{x-2}}{4!} + \frac{(t^5 - 5t^3 + 4t)\Delta^5 l_{x-2}}{5!} \quad (25)$$

$$l_{x+t} = \frac{\Delta^0 l_x}{0!} + \frac{t\Delta^1 l_x}{1!} + \frac{(t^2 - t)\Delta^2 l_{x-1}}{2!} + \frac{(t^3 - t)\Delta^3 l_{x-1}}{3!} + \frac{(t^4 - t^2 - 2t^3 + 2t)\Delta^4 l_{x-2}}{4!} + \frac{(t^5 - 5t^3 + 4)\Delta^5 l_{x-2}}{5!} + \ldots \quad (26)$$

Differentiate the survival function l_{x+t} with respect to t and obtain

$$\frac{\mathrm{d}l_{x+t}}{dt} = \frac{\Delta^1 l_x}{1!} + \frac{(2t-1)\Delta^2 l_{x-1}}{2!} + \frac{(3t^2-1)\Delta^3 l_{x-1}}{3!} + \frac{(4t^3 - 2t - 6t^2 + 2)\Delta^4 l_{x-2}}{4!} + \frac{(5t^4 - 15t^2 + 4)\Delta^5 l_{x-2}}{5!} + \ldots \quad (27)$$

By definition,

$$\mu_{x+t} = -\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt} \quad (28)$$

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{\Delta^1 l_x}{l_x} - \frac{(2t-1)\Delta^2 l_{x-1}}{2l_x} - \frac{(3t^2-1)\Delta^3 l_{x-1}}{3!} - \frac{(4t^3 - 2t - 6t^2 + 2)\Delta^4 l_{x-2}}{4!l_x} - \frac{(5t^4 - 15t^2 + 4)\Delta^5 l_{x-2}}{5!l_x} \quad (29)$$

It is required to set $t = 0$ in (29) to have

$$\mu_x = \left[-\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt}\right]_{t=0} = -\frac{1}{l_x}\frac{dl_x}{dx} \quad (30)$$

Setting $t = 0$ in (29)

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{\Delta^1 l_x}{l_x} - \frac{(2(0)-1)\Delta^2 l_{x-1}}{2l_x} - \frac{(3(0)^2-1)\Delta^3 l_{x-1}}{3!} - \frac{(4^3(0) - 2(0) - 6(0)^2 + 2)\Delta^4 l_{x-2}}{4!l_x} - \frac{(5(0)^4 - 15(0)^2 + 4)\Delta^5 l_{x-2}}{5!l_x} \quad (31)$$

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$$-\frac{1}{l_x}\frac{dl_x}{dx}=-\frac{\Delta^1 l_x}{l_x}+\frac{1}{2l_x}\Delta^2 l_{x-1}+\frac{1}{3!}\Delta^3 l_{x-1}-\frac{2}{4!l_x}\Delta^4 l_{x-2}-\frac{4}{5!l_x}\Delta^5 l_{x-2}+\ldots \tag{32}$$

$$\Delta^n l_x=\sum_{k=0}^{n}(-1)^k\frac{n!l_x+(n-k)}{(n-k)!k!} \tag{33}$$

$r=0,1,2,3,\ldots$

$$\Delta^1 l_x=l_{x+1}-l_x \tag{34}$$

$$\Delta^2 l_x=\Delta l_{x+1}-\Delta l_x \tag{35}$$

$$\Delta^2 l_x=l_{x+2}-l_{x+1}-(l_{x+1}-l_x) \tag{36}$$

$$\Delta^2 l_x=l_{x+2}-l_{x+1}-l_{x+1}+l_x \tag{37}$$

$$\Delta^2 l_x=l_{x+2}-2l_{x+1}+l_x \tag{38}$$

$$\Delta^2 l_{x-1}=l_{x+1}-l_x-l_x+l_{x-1} \tag{39}$$

$$\Delta^2 l_{x-1}=l_{x+1}-2l_x+l_{x-1} \tag{40}$$

$$\Delta l_{x-2}=l_{x-1}-l_{x-2} \tag{41}$$

$$\Delta^2 l_{x-2}=l_x-l_{x-1}-(l_{x-1}-l_{x-2}) \tag{42}$$

$$\Delta^2 l_{x-2}=l_x-l_{x-1}-(l_{x-1}-l_{x-2}) \tag{43}$$

$$\Delta^2 l_{x-2}=l_x-l_{x-1}-l_{x-1}+l_{x-2} \tag{44}$$

$$\Delta^2 l_{x-2}=l_x-2l_{x-1}+l_{x-2} \tag{45}$$

$$\Delta^3 l_{x-2}=\Delta l_x-2\Delta l_{x-1}+\Delta l_{x-2} \tag{46}$$

$$\Delta^3 l_{x-2}=l_{x+1}-l_x-2(l_x-l_{x-1})+l_{x-1}-l_{x-2} \tag{47}$$

$$\Delta^3 l_{x-2}=l_{x+1}-l_x-2l_x+2l_{x-1}+l_{x-1}-l_{x-2} \tag{48}$$

$$\Delta^3 l_{x-2}=l_{x+1}-3l_x+3l_{x-1}-l_{x-2} \tag{49}$$

$$\Delta^4 l_{x-2}=\Delta l_{x+1}-3\Delta l_x+3\Delta l_{x-1}-\Delta l_{x-2} \tag{50}$$

$$\Delta^4 l_{x-2}=l_{x+2}-l_{x+1}-3(l_{x+1}-l_x)+l_x-l_{x-1}-(l_{x-1}-l_{x-2}) \tag{51}$$

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$$\Delta^4 l_{x-2} = l_{x+2} - l_{x+1} - 3l_{x+1} + 3l_x + l_x - l_{x-1} - l_{x-1} + l_{x-2} \tag{52}$$

$$\Delta^4 l_{x-2} = l_{x+2} - 4l_{x+1} + 6l_x - 4l_{x-1} + l_{x-2} \tag{53}$$

$$\Delta^5 l_{x-2} = \Delta l_{x+2} - 4\Delta l_{x+1} + \Delta 6l_x - 4\Delta l_{x-1} + \Delta l_{x-2} \tag{54}$$

$$\Delta^5 l_{x-2} = l_{x+3} - l_{x+2} - 4(l_{x+2} - l_{x+1}) + 6(l_{x+1} - l_x) - 4(l_x - l_{x-1}) + l_{x-1} - l_{x-2} \tag{55}$$

$$\Delta^5 l_{x-2} = l_{x+3} - l_{x+2} - 4l_{x+2} + 4l_{x+1} + 6l_{x+1} - 6l_x - 4l_x + 4l_{x-1} + l_{x-1} - l_{x-2} \tag{56}$$

$$\Delta^5 l_{x-2} = l_{x+3} - 5l_{x+2} + 10l_{x+1} - 10l_x + 5l_{x-1} - l_{x-2} \tag{57}$$

$$\begin{aligned}\mu_x = \frac{-\frac{dl_x}{dx}}{l_x} &= -\frac{(l_{x+1} - l_x)}{l_x} + \frac{1}{2l_x}(l_{x+1} - 2l_x + l_{x-1}) \\ &+ \frac{1}{6l_x}(l_{x+2} - 3l_{x+1} + 3l_x - l_{x-1}) \\ &- \frac{1}{12l_x}(l_{x+2} - 4l_{x+1} + 6l_x - 4l_{x-1} + l_{x-2}) \\ &- \frac{1}{30l_x}(l_{x+3} - 5l_{x+2} + 10l_{x+1} - 10l_x + 5l_{x-1} - l_{x-2})\end{aligned} \tag{58}$$

$$\begin{aligned}\mu_x = \frac{-\frac{dl_x}{dx}}{l_x} &= -\frac{(l_{x+1} - l_x)}{l_x} + \frac{(l_{x+1} - 2l_x + l_{x-1})}{2l_x} \\ &+ \frac{(l_{x+2} - 3l_{x+1} + 3l_x - l_{x-1})}{6l_x} \\ &- \frac{(l_{x+2} - 4l_{x+1} + 6l_x - 4l_{x-1} + l_{x-2})}{12l_x} \\ &- \frac{(l_{x+3} - 5l_{x+2} + 10l_{x+1} - 10l_x + 5l_{x-1} - l_{x-2})}{30l_x}\end{aligned} \tag{59}$$

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$$\mu_x = \frac{-\frac{dl_x}{dx}}{l_x} = \frac{-l_{x+1}+l_x}{l_x} + \frac{l_{x+1}-2l_x+l_{x-1}}{2l_x} + \frac{l_{x+2}-3l_{x+1}+3l_x-l_{x-1}}{6l_x} \quad \frac{-l_{x+2}+4l_{x+1}-6l_x+4l_{x-1}-l_{x-2}}{12l_x} \quad \frac{-l_{x+3}+5l_{x+2}-10l_{x+1}+10l_x-5l_{x-1}+l_{x-2}}{30l_x} \tag{60}$$

$$\mu_x = \frac{\begin{array}{l} -60l_{x+1}+60l_x+30l_{x+1}-60l_x+30l_{x-1} \\ +10l_{x+2}-30l_{x+1}+30l_x-10l_{x-1} \\ -5l_{x+2}-20l_{x+1}-30l_x+20l_{x-1}-5l_{x-2} \\ -2l_{x+3}+10l_{x+2}-20l_{x+1}+20l_x-10l_{x-1}+2l_{x-2} \end{array}}{60l_x} \tag{61}$$

$$\mu_x = \frac{20l_x-60l_{x+1}+15l_{x+2}-2l_{x+3}+30l_{x-1}-3l_{x-2}}{60l_x} \tag{62}$$

$$\mu_{x+t} = \frac{20l_{x+t}-60l_{x+t+1}+15l_{x+t+2}-2l_{x+t+3}+30l_{x+t-1}-3l_{x+t-2}}{60l_{x+t}} \tag{63}$$

$$_t p_x = e^{-\int_0^t \mu_{x+s} ds} \tag{64}$$

$$_t p_x = e^{-\int_0^t \left\{\frac{\left[20l_{x+s}-60l_{x+s+1}+15l_{x+s+2}-2l_{x+s+3}+30l_{x+s-1}-3l_{x+s-2}\right]}{60l_{x+s}}\right\} ds} \tag{65}$$

The equation (63) has the following implication as stated in the theorem below

Theorem:

Let T_x be the random life time of a life aged x, then

$$\int_0^\infty \xi f_{T_x}(\xi) d\xi = \frac{1}{2} + \sum_{s=1}^{\infty} {}_s p_x - \frac{1}{12}\left(\frac{20l_x-60l_{x+1}+15l_{x+2}-2l_{x+3}+30l_{x-1}-3l_{x-2}}{60l_x}\right) - \frac{1}{720}\frac{1}{l_x}\frac{d^3}{dx^3}l_x \tag{66}$$

Proof

$$\mathbf{E}(T_x) = \int_0^\infty \xi f_{T_x}(\xi) d\xi = \int_0^\infty \xi \times \left({}_\xi p_x\right)\mu_{x+\xi} d\xi \tag{67}$$

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$$\mathbf{E}(T_x)=\int_0^{\infty}\xi\frac{d}{d\xi}\left(-_{\xi}p_x\right)d\xi \tag{63}$$

$$\mathbf{E}(T_x)=\int_0^{\infty}\xi\frac{d}{d\xi}\left(-_{\xi}p_x\right)d\xi=-\left\{\left[\xi\times_{\xi}p_x\right]_0^{\infty}-\int_0^{\infty}\left({}_{\xi}p_x\right)d\xi\right\} \tag{68}$$

$$\mathbf{E}(T_x)=\int_0^{\infty}\left({}_{\xi}p_x\right)d\xi \tag{69}$$

$$\mathbf{E}(T_x)=\int_0^{\infty}\left({}_{\xi}p_x\right)d\xi \tag{70}$$

$$\int_{\xi}^{\xi+1}l_{x+s}ds=\int_0^1 l_{x+s+u}du \tag{71}$$

$$l_{x+s+u}=l_{x+s}+u\frac{d}{ds}l_{x+s}+\frac{u^2}{2}\frac{d^2}{ds^2}l_{x+s}+\frac{u^3}{6}\frac{d^3}{ds^3}l_{x+s}+\frac{u^4}{24}\frac{d^4}{ds^4}l_{x+s}+\frac{u^5}{120}\frac{d^5}{ds^5}l_{x+s}+\ldots \tag{72}$$

$$\int_0^1 l_{x+s+u}du=\int_0^1\left(l_{x+s}+u\frac{d}{ds}l_{x+s}+\frac{u^2}{2}\frac{d^2}{ds^2}l_{x+s}+\frac{u^3}{6}\frac{d^3}{ds^3}l_{x+s}+\frac{u^4}{24}\frac{d^4}{ds^4}l_{x+s}+\frac{u^5}{120}\frac{d^5}{ds^5}l_{x+s}+\ldots\right)du \tag{73}$$

$$\int_0^1 l_{x+s+u}du=\left[l_{x+s}u+\frac{u^2}{2}\frac{d}{ds}l_{x+s}+\frac{u^3}{6}\frac{d^2}{ds^2}l_{x+s}+\frac{u^4}{24}\frac{d^3}{ds^3}l_{x+s}+\frac{u^5}{120}\frac{d^4}{ds^4}l_{x+s}+\frac{u^6}{720}\frac{d^5}{ds^5}l_{x+s}+\ldots\right]_0^1 \tag{74}$$

$$\int_0^1 l_{x+s+u}du=l_{x+s}+\frac{1}{2}\frac{d}{ds}l_{x+s}+\frac{1}{6}\frac{d^2}{ds^2}l_{x+s}+\frac{1}{24}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{120}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{720}\frac{d^5}{ds^5}l_{x+s}+\ldots \tag{75}$$

$$l_{x+s+1}=l_{x+s}+\frac{d}{ds}l_{x+s}+\frac{1}{2}\frac{d^2}{ds^2}l_{x+s}+\frac{1}{6}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{24}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{120}\frac{d^5}{ds^5}l_{x+s}+\ldots \tag{76}$$

$$\frac{d}{ds}l_{x+s+1}=\frac{d}{ds}l_{x+s}+\frac{d^2}{ds^2}l_{x+s}+\frac{1}{2}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{6}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{24}\frac{d^5}{ds^5}l_{x+s}+\frac{1}{120}\frac{d^6}{ds^6}l_{x+s}\ldots \tag{77}$$

$$\frac{d}{ds}l_{x+s+1}-\frac{d}{ds}l_{x+s}=\frac{d^2}{ds^2}l_{x+s}+\frac{1}{2}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{6}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{24}\frac{d^5}{ds^5}l_{x+s}+\frac{1}{120}\frac{d^6}{ds^6}l_{x+s}\ldots \tag{78}$$

$$\frac{d^2}{ds^2}l_{x+s+1}=\frac{d^2}{ds^2}l_{x+s}+\frac{d^3}{ds^3}l_{x+s}+\frac{1}{2}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{6}\frac{d^5}{ds^5}l_{x+s}+\frac{1}{24}\frac{d^6}{ds^6}l_{x+s}+\frac{1}{120}\frac{d^7}{ds^7}l_{x+s}\ldots \tag{79}$$

$$\frac{d^3}{ds^3}l_{x+s+1}=\frac{d^3}{ds^3}l_{x+s}+\frac{d^4}{ds^4}l_{x+s}+\frac{1}{2}\frac{d^5}{ds^5}l_{x+s}+\frac{1}{6}\frac{d^6}{ds^6}l_{x+s}+\frac{1}{24}\frac{d^7}{ds^7}l_{x+s}+\frac{1}{120}\frac{d^8}{ds^8}l_{x+s}\ldots \tag{80}$$

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$$\frac{d^3}{ds^3}l_{x+s+1}-\frac{d^3}{ds^3}l_{x+s}=\frac{d^4}{ds^4}l_{x+s}+\frac{1}{2}\frac{d^5}{ds^5}l_{x+s}+\frac{1}{6}\frac{d^6}{ds^6}l_{x+s}+\frac{1}{24}\frac{d^7}{ds^7}l_{x+s}+\frac{1}{120}\frac{d^8}{ds^8}l_{x+s}\cdots \quad (81)$$

Now,

$$\frac{1}{2}\left(l_{x+s}+l_{x+s+1}\right)=\frac{1}{2}\begin{bmatrix} l_{x+s}+l_{x+s}+\frac{d}{ds}l_{x+s}+\frac{1}{2}\frac{d^2}{ds^2}l_{x+s}+\frac{1}{6}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{24}\frac{d^4}{ds^4}l_{x+s} \\ +\frac{1}{120}\frac{d^5}{ds^5}l_{x+s}+\ldots \end{bmatrix} \quad (82)$$

$$\begin{aligned}&\frac{1}{2}\left(l_{x+s}+l_{x+s+1}\right)\\ &=\left[l_{x+s}+\frac{1}{2}\frac{d}{ds}l_{x+s}+\frac{1}{4}\frac{d^2}{ds^2}l_{x+s}+\frac{1}{12}\frac{d^3}{ds^3}l_{x+s}+\frac{1}{48}\frac{d^4}{ds^4}l_{x+s}+\frac{1}{240}\frac{d^5}{ds^5}l_{x+s}+\ldots\right]\end{aligned} \quad (83)$$

$$\int_{\xi}^{\xi+1}l_{x+s}ds=\frac{1}{2}\left(l_{x+s}+l_{x+s+1}\right)-\frac{1}{12}\left(\frac{d}{ds}l_{x+s+1}-\frac{d}{ds}l_{x+s}\right)+\frac{1}{720}\left(\frac{d^3}{ds^3}l_{x+s+1}-\frac{d^3}{ds^3}l_{x+s}\right)-\ldots \quad (84)$$

Letting $s=0$ and $\xi=0$

$$\int_{0}^{1}l_{x+s}ds=\frac{1}{2}\left(l_{x}+l_{x+1}\right)-\frac{1}{12}\left(\frac{d}{ds}l_{x+1}-\frac{d}{ds}l_{x}\right)+\frac{1}{720}\left(\frac{d^3}{ds^3}l_{x+1}-\frac{d^3}{ds^3}l_{x}\right)-\ldots \quad (85)$$

When $s=1$ and $\xi=1$

$$\int_{1}^{2}l_{x+s}ds=\frac{1}{2}\left(l_{x+1}+l_{x+2}\right)-\frac{1}{12}\left(\frac{d}{ds}l_{x+2}-\frac{d}{ds}l_{x+1}\right)+\frac{1}{720}\left(\frac{d^3}{ds^3}l_{x+2}-\frac{d^3}{ds^3}l_{x+1}\right)-\ldots \quad (86)$$

When $s=2$ and $\xi=2$

$$\int_{2}^{3}l_{x+s}ds=\frac{1}{2}\left(l_{x+2}+l_{x+3}\right)-\frac{1}{12}\left(\frac{d}{ds}l_{x+3}-\frac{d}{ds}l_{x+2}\right)+\frac{1}{720}\left(\frac{d^3}{ds^3}l_{x+3}-\frac{d^3}{ds^3}l_{x+2}\right)-\ldots \quad (87)$$

Sum up the integrals in (84)-(87) to get.

$$\int_{0}^{\infty}l_{x+s}ds=\frac{1}{2}l_x+\sum_{s=1}^{\infty}l_{x+s}+\frac{1}{12}\frac{d}{dx}l_x-\frac{1}{720}\frac{d^3}{dx^3}l_x+\ldots \quad (88)$$

$$\int_{0}^{\infty}l_{x+s}ds=\frac{1}{2}l_x+\sum_{s=1}^{\infty}l_{x+s}-\frac{1}{12}l_x\mu_x-\frac{1}{720}\frac{d^3}{dx^3}l_x+\ldots \quad (89)$$

$$\int_{0}^{\infty}\frac{l_{x+s}}{lx}ds=\frac{1}{l_x}\left(\frac{1}{2}l_x+\sum_{s=1}^{\infty}l_{x+s}-\frac{1}{12}l_x\mu_x-\frac{1}{720}\frac{d^3}{dx^3}l_x+\ldots\right) \quad (90)$$

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Equate (70) and (90) to obtain the expected value of the random life time

$$\int_0^\infty {}_s p_x ds = \mathbf{E}\left(T_x\right) = \frac{1}{2} + \sum_{s=1}^{\infty} {}_s p_x - \frac{1}{12}\left(\frac{20l_x - 60l_{x+1} + 15l_{x+2} - 2l_{x+3} + 30l_{x-1} - 3l_{x-2}}{60l_x}\right) - \frac{1}{720}\frac{1}{l_x}\frac{d^3}{dx^3}l_x \tag{91}$$

QED

3. Result and Discussion

The results obtained are given in Tables 1 and 2 as well as in Figure 1. One of the key features of the Gauss forward model is its potential to incorporate random fluctuations or kinks in mortality rates that are difficult to predict deterministically as observed in Figure 1(f). This approach recognizes that while mortality rates may follow broad trends, there can be unexpected changes due to external factors like pandemics, economic shifts or healthcare advancements as observed in Figure 1(b). The exhibited random potential helps in accounting for uncertainty and provides a range of potential outcomes of the deterministic forecast. This is particularly useful in risk management where insurers and pension funds need to consider worst-case and best-case scenarios in their financial planning. Figures 1(b) and (f) may have the following consequences: (i) the wavy kinks may lead to inaccurate mortality predictions as the model may struggle to capture the underlying patterns and trends, (ii) the wavy kinks can result in biased life expectancy estimates which can affect calculations of life expectancy, health life expectancy, and disability-free life expectancy, (iii) the wavy kinks can increase uncertainty in mortality modelling, making it more challenging to predict mortality rates and life expectancy, (iv) the wavy kinks can lead to model risks as mortality models may not be able to capture the underlying patterns and trends, potentially leading to incorrect conclusions and decisions, (v) the wavy kinks can have significant actuarial implications, affecting the calculations of insurance premiums, pension benefits, and social security payments.

Table 1: Gauss forward mortality table for male

x	l_x	μ_x	$l_x\mu_x$	d_x
0	100000	0.000054	5	587
1	99413	0.000252	25	40
2	99373	0.000194	19	27
3	99346	0.000166	16	23
4	99323	0.000148	15	18
5	99306	0.000141	14	16
6	99290	0.000140	14	15
7	99276	0.000114	11	14
8	99262	0.000095	9	12
9	99249	0.000083	8	10
10	99239	0.000114	11	9
11	99230	0.000192	19	10
12	99221	0.000325	32	15
13	99206	0.000476	47	25
14	99181	0.000641	64	40

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15	99141	0.033941	3365	56
16	99086	0.247959	24569	70
17	99015	0.997167	98734	84
18	98932	0.331255	32772	94
19	98837	0.497806	49202	103
20	99	51.196096	5055	112
21	98623	0.001252	124	120
22	98503	0.001225	121	124
23	98379	0.001161	114	123
24	98257	0.001107	109	118
25	98139	0.001072	105	111
26	98028	0.001051	103	107
27	97921	0.001087	106	104
28	97817	0.001134	111	105
29	97713	0.001179	115	108
30	97604	0.001241	121	113
31	97491	0.001329	130	118
32	97373	0.001416	138	125
33	97248	0.001538	150	134
34	97114	0.001664	162	144
35	96971	0.001809	175	155
36	96815	0.001963	190	168
37	96647	0.002141	207	183
38	96464	0.002331	225	198
39	96266	0.002528	243	216
40	96050	0.002733	262	234
41	95816	0.002970	285	253
42	95563	0.003209	307	274
43	95290	0.003462	330	295
44	94994	0.003758	357	318
45	94676	0.004033	382	343
46	94333	0.004257	402	370
47	93963	0.004450	418	392
48	93571	0.004637	434	410
49	93161	0.004862	453	426
50	92735	0.005158	478	443
51	92292	0.005533	511	465
52	91827	0.005999	551	494
53	91333	0.006550	598	530
54	90803	0.007162	650	574
55	90229	0.007827	706	624
56	89605	0.008558	767	678
57	88927	0.009352	832	736
58	88191	0.010219	901	799

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59	87392	0.011192	978	866
60	86526	0.012289	1063	939
61	85587	0.013506	1156	1021
62	84567	0.014840	1255	1109
63	83458	0.016332	1363	1205
64	82253	0.017961	1477	1308
65	80945	0.019712	1596	1419
66	79525	0.021554	1714	1536
67	77989	0.023486	1832	1655
68	76334	0.025624	1956	1773
69	74561	0.028067	2093	1893
70	72668	0.030648	2227	2024
71	70645	0.033248	2349	2161
72	68484	0.035911	2459	2289
73	66195	0.038862	2573	2405
74	63790	0.042292	2698	2515
75	61275	0.046202	2831	2634
76	58641	0.050501	2961	2765
77	55877	0.055274	3089	2896
78	52980	0.060584	3210	3025
79	49955	0.066461	3320	3150
80	46805	0.072961	3415	3266
81	43539	0.080077	3486	3369
82	40170	0.087756	3525	3453
83	36717	0.095979	3524	3509
84	33208	0.104560	3472	3528
85	29680	0.113456	3367	3502
86	26177	0.122621	3210	3425
87	22753	0.131822	2999	3293
88	19460	0.141027	2744	3108
89	16351	0.150114	2455	2875
90	13476	0.158828	2140	2602
91	10874	0.167145	1818	2299
92	8575	0.175009	1501	1980
93	6596	0.181620	1198	1658
94	4938	0.186648	922	1348
95	3590	0.190311	683	1057
96	2533	0.192808	488	799
97	1734	0.194002	336	582
98	1152	0.194285	224	409
99	743	0.195873	146	277
100	466	0.196280	91	182
101	284	0.194542	55	117
102	167	0.196307	33	72

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103	95	0.185789	18	43
104	52	0.197756	10	25
105	27	0.181481	5	14
106	14	0.164286	2	7
107	6	0.255556	2	4
108	3	0.116667	0	2
109	1	0.050000	0	1
110	0	0.000000	0	0

Table 2: Gauss forward mortality table for female

x	l_x	μ_x	$l_x\mu_x$	d_x
0	100000	0.000057	6	495
1	99505	0.000198	20	35
2	99470	0.000138	14	23
3	99447	0.000118	12	17
4	99430	0.000120	12	13
5	99418	0.000122	12	12
6	99406	0.000117	12	12
7	99394	0.000104	10	12
8	99382	0.000100	10	11
9	99371	0.000100	10	10
10	99361	0.000099	10	10
11	99351	0.000135	13	10
12	99341	0.000183	18	12
13	99330	0.000249	25	16
14	99314	0.000307	31	21
15	99293	0.000361	36	28
16	99265	0.000397	39	33
17	99232	0.033638	3338	38
18	99194	0.248943	24694	40
19	99154	0.998191	98975	40
20	99114	0.332315	32937	40
21	99074	0.498875	49426	41
22	99074	50.396296	4989	41
23	98992	0.000438	43	42
24	98950	0.000455	45	43
25	98907	0.000474	47	44
26	98863	0.000498	49	46
27	98817	0.000538	53	48
28	98769	0.000568	56	51
29	98718	0.000619	61	55
30	98663	0.000672	66	59

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31	98605	0.000731	72	64
32	98541	0.000820	81	69
33	98472	0.000895	88	76
34	98396	0.000984	97	84
35	98311	0.001090	107	93
36	98219	0.001185	116	102
37	98117	0.001282	126	111
38	98005	0.001390	136	121
39	97884	0.001513	148	131
40	97753	0.001633	160	142
41	97611	0.001744	170	154
42	97457	0.001842	180	165
43	97292	0.001867	182	175
44	97117	0.002587	251	184
45	96933	0.000141	14	195
46	96739	0.003038	294	207
47	96531	0.003551	343	221
48	96511	0.002584	249	235
49	96076	0.002910	280	251
50	95825	0.003158	303	269
51	95556	0.003461	331	291
52	95265	0.003819	364	316
53	94949	0.004222	401	347
54	94602	0.004670	442	382
55	94220	0.005171	487	421
56	93799	0.005699	535	464
57	93335	0.006289	587	511
58	92824	0.006929	643	561
59	92264	0.007628	704	614
60	91649	0.008420	772	673
61	90976	0.009287	845	738
62	90239	0.010213	922	808
63	89431	0.011251	1006	883
64	88548	0.012413	1099	964
65	87585	0.013639	1195	1052
66	86533	0.014938	1293	1147
67	85386	0.016280	1390	1243
68	84143	0.017792	1497	1341
69	82801	0.019525	1617	1442
70	81359	0.021375	1739	1556
71	79803	0.023289	1859	1678
72	78125	0.025277	1975	1799
73	76326	0.027515	2100	1917
74	74409	0.030147	2243	2036

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75	72373	0.033135	2398	2170
76	70203	0.036407	2556	2320
77	67883	0.040021	2717	2477
78	65406	0.044141	2887	2636
79	62770	0.048893	3069	2801
80	59969	0.054134	3246	2977
81	56992	0.059763	3406	3158
82	53833	0.065892	3547	3328
83	50505	0.072597	3667	3479
84	47027	0.079981	3761	3609
85	43418	0.088093	3825	3716
86	39702	0.096836	3845	3796
87	35906	0.106189	3813	3839
88	32067	0.115913	3717	3834
89	28234	0.125903	3555	3770
90	24463	0.135955	3326	3642
91	20822	0.145866	3037	3446
92	17376	0.155536	2703	3186
93	14190	0.164340	2332	2873
94	11317	0.171918	1946	2520
95	8797	0.178163	1567	2139
96	6658	0.183206	1220	1755
97	4903	0.186736	916	1390
98	3513	0.189866	667	1064
99	2449	0.193290	473	786
100	1663	0.195801	326	566
101	1097	0.196961	216	396
102	701	0.198407	139	268
103	433	0.198460	86	176
104	258	0.197545	51	111
105	147	0.193424	28	67
106	80	0.188750	15	39
107	41	0.180894	7	21
108	20	0.154167	3	11
109	9	0.135185	1	5
110	4	0.283333	1	2
111	2	0.200000	0	1
112	1	0.050000	0	0
113	0	0.000000	0	0

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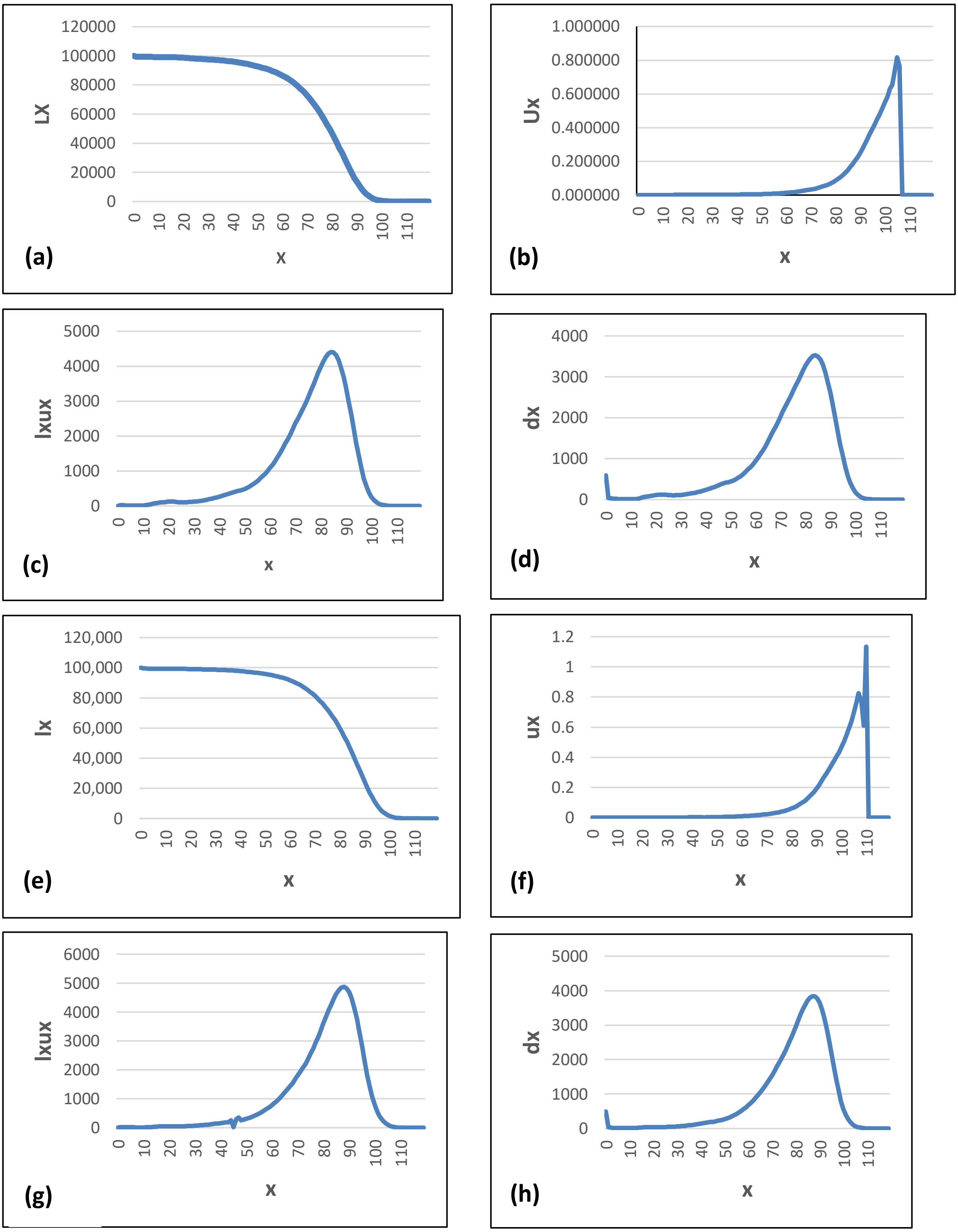

Figure 1. (a) The trajectory of male survival function. **(b)** The trajectory of male mortality rate intensity. **(c)** The trajectory of male curve of death. **(d)** The trajectory of the male number of deaths. **(e)** The trajectory of female survival function. **(f)** The trajectory of female curve of death. **(g)** The trajectory of female mortality rate intensity. **(h)** The trajectory of female number of deaths.

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A higher l_x at older ages compared to the current population indicates mortality improvement over time. This would suggest that individuals are expected to live longer as improvements in healthcare, lifestyle and technology reduce death rates. In Figure 1(a) and (e), the l_x curve flattens or declines suggesting that the improvements in survival are slowing down or even reversing. Where the Gauss forward interpolation produces a decreasing μ_x over time, this indicates a decline in mortality rates, suggesting that people are living longer and mortality is improving.

In Tables 1 and 2, the constant or increasing μ_x would suggest stable or deteriorating mortality, meaning life expectancy is not improving or is expected to decrease. Gauss interpolation is used to estimate mortality rates at intermediate ages or time points, based on a set of observed mortality rates. This allows for the creation of a mortality model that can be used to predict future mortality rates. The interpolation has a smoothing effect on the mortality rates, which can help to reduce the impact of random fluctuations in the data. This can result in a more stable and reliable mortality model. Consequently, the interpolation can handle sparse data, which is common in mortality modelling. This means that the approach can be used to estimate mortality rates even when there is limited data available. In Figure 1(c) and (g), the modal age at death is roughly 84 which is normal for human beings. The trajectories form asymptotes to age axis while Figure 1(d) and (h) show the trajectories for the number of deaths.

The Gauss forward model often produces a range of mortality rates under different assumptions (different mortality improvement scenarios). This is to allow actuaries to conduct stress testing to understand how sensitive their projections fall in line with changes in mortality trends. The ability to model different future scenarios helps to understand risks such as how much life expectancy could deviate from current projections in case of unforeseen events like pandemics or technological breakthroughs. The wavy kinks in the female's curves Figure 6 suggest that the Gauss forward interpolation provides a random approach to mortality which can be used to assess solvency risk and ensure that life insurers and pension funds maintain adequate reserves in light of future uncertainties in mortality trends.

It is recommended that Governments and health organizations can use the projections from Gauss forward interpolation to plan for healthcare resources, allocate funds and design public health policies aimed at addressing aging populations and improving life expectancy. Accurate mortality projections are crucial for pricing life insurance, annuities and pension plans. If mortality is improving more than expected, insurance companies may face increased payouts for annuities and pension funds. Consequently, insurers may need to adjust their reserves to account for longer lifespans.

4. Conclusions

Following the results of this study, the forward model can be integrated with other actuarial models to assess how changes in mortality rates may affect the pricing and risk of life insurance or annuities. It can be combined with models that take into account interest rates, inflation and investment returns allowing for more comprehensive financial modeling. This integration provides a better understanding of how changes in mortality may impact future liabilities, helping financial institutions to set more accurate reserves and ensure solvency. It may further allow actuaries to adjust their models in response to evolving market conditions or demographic changes.

While the forward model can provide more accurate projections of mortality, its effectiveness is highly sensitive to the assumptions underlying it. The model assumes that mortality improvements may continue at the same rate as in the past where future improvements are slower than expected or vice versa, this could lead to significant discrepancies in the predicted mortality rates. Therefore, a careful calibration of the model is essential as small changes in assumptions about the rate of improvement applied can lead to large differences in the output. This requires careful selection of input mortality data and an understanding of the potential impacts of various assumptions on long-term projections.

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The forward model is especially useful for modeling mortality rates over long-time horizons. This is crucial for financial products like life insurance policies, pensions and annuities, where the future liabilities are long-term and highly sensitive to mortality assumptions. Accurate mortality predictions can affect the pricing and risk management strategies of these products. By using a forward-looking model like the forward, insurers and pension funds can estimate their liabilities better which helps in setting appropriate premiums, reserves and solvency buffers.

The forward model generally operates on population-level mortality data such as national mortality tables or life expectancy projections. These data can be influenced by various factors including changes in public health policies, the advent of new treatments or shifts in demographic patterns. The model may struggle to account for individual heterogeneity such as differences in mortality rates between socioeconomic groups. This can be a limitation when applying the model to highly personalized products like individual life insurance policies or tailored annuity contracts. Calibrating the forward model to historical mortality data can be challenging, especially when mortality trends exhibit non-linear behaviors or when data quality is inconsistent. Furthermore, the model might need to be adapted to reflect specific market or demographic conditions which can introduce further complexity. There is a risk that the model might not capture future mortality dynamics accurately if the calibration process does not fully account for shifts in underlying causes of death such as the outbreak of pandemics or technological breakthroughs in healthcare.

The forward model assumes that mortality trends are predictable based on historical data but it can fail to account for sudden social, environmental or policy changes. For instance, mortality rates might improve drastically due to a new public health intervention or conversely, a sudden rise in mortality could occur due to an economic or environmental crisis. Relying too heavily on the forward model may lead to overconfidence in predictions, potentially exposing insurers or pension funds to significant risks. Furthermore, incorrect predictions about future mortality rates could have social implications, especially if they impact retirement planning or healthcare policies.

The forward model provides a powerful tool for projecting mortality rates to imply improved forecasting accuracy, better integration with financial modeling and a more comprehensive understanding of future mortality risks. However, it requires careful calibration, validation and consideration of various assumptions as small errors in input data or assumptions can lead to significant inaccuracies. Furthermore, the model may be integrated with other models to ensure that it reflects the full range of uncertainties surrounding future mortality trends. The directions for further research may require that the stochastic nature of the model may also involve additional validation and stress testing to ensure that the model remains within acceptable risk limits.

Future research should focus on penalized splines which can accommodate a combination of spline smoothing and penalty functions to control the smoothness of the mortality curve. Furthermore, actuaries can experiment Bayesian smoothing to smooth out the mortality curve and then capture the uncertainty associated with wavy kinks. The use of complex sophisticated interpolation models like the forward to estimate mortality rates must be accompanied by rigorous regulatory scrutiny. Regulatory bodies often require actuaries to demonstrate that their models are sound, transparent and based on reliable data. Life insurers need to ensure that their use of the forward model aligns with regulatory requirements and best practices in actuarial science. This could involve regular validation against actual mortality outcomes and maintaining transparency in the assumptions used in the model.

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