Iconic Computational Strategy for Measuring Actuarial Present Value of Benefits Under the Extended Makeham's Function

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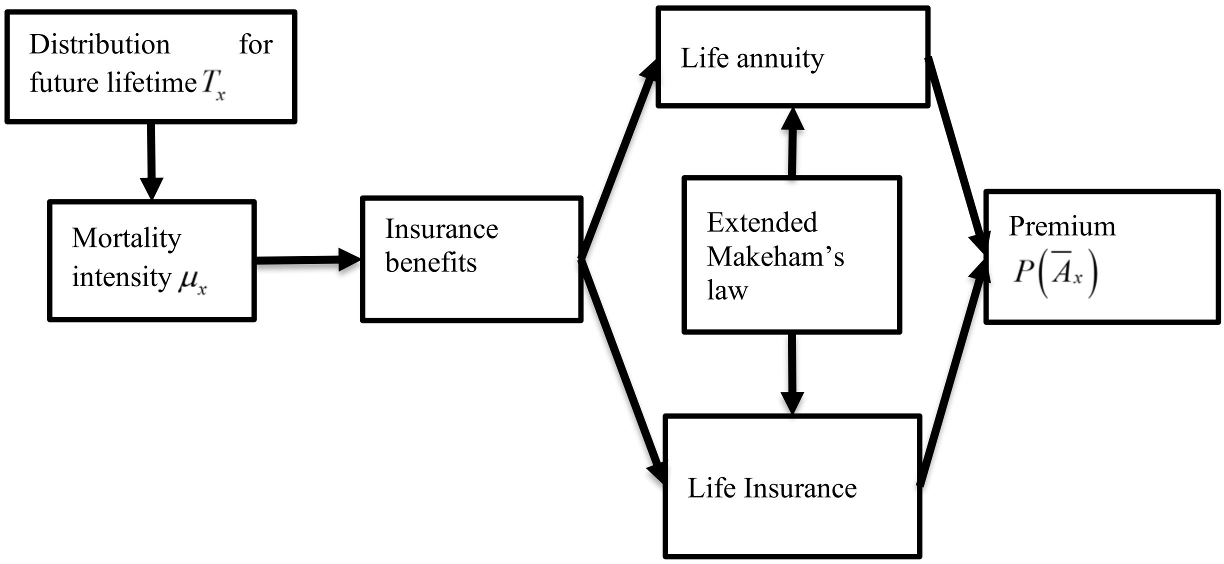

Abstract

This paper initiates the extended Makeham's function deploying rigorous analytic constructions with interesting results for the design of life insurance benefits. The information reinforcing the actuarial valuation of life insurance benefits is dependent on substantiated contingencies and underwriting transactions without reflecting the expected future trends. The objective is to employ refinements to existing methods to capture mortality rate intensities accurately for actuarial valuation. The theoretical underpinnings of instantaneous mortality rate model were explored by conducting deep analytic derivations of model equations through numerical discretisation to examine the degree to which the functional equations impact the mortality-dependant and the actuarial valuation of life insurance benefits. The procedure employed numerical discretisation to prove the benefit function theorem which pricing actuaries depend upon to derive the net premiums. Computational evidence from the results, reveal that the benefit functions exhibit full temporal dynamics of life insurance valuation satisfying the duality principle. The trajectories of both benefit functions also confirm monotonicity and smoothness but the smooth evolution implies actuarial coherence and continuity of the underlying mortality intensity and force of interest. The monotonic behaviour demonstrates the trade-off between longevity risk and mortality risk which are essential to insurance products design. The continuous-discrete relationship demonstrates analytic continuum which serves as iconic bridge between theoretical and applied life insurance practice. Since the trajectories of life annuity and insurance functions both satisfy the duality principle, it is therefore recommended that the law be employed by actuaries when conducting actuarial valuations of life insurance schemes.

Keywords: *Contingencies, discretisation, duality, refinement, mortality*

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1.0 Introduction

The estimation of the present value of benefits based on governing laws at an acceptable degree of accuracy has aroused keen interest in life insurance practice. The present value of the annuity defines the amount which the scheme sets aside now to fund the future obligations. The estimation of life annuities in actuarial valuations applies assumptions on the annuitant's life expectancy and mortality rates. These assumptions are mostly dependent on empirical life tables, which compute the average number of years a life is expected to survive beyond retirement. These mortality tables are usually adjusted for factors like age and gender. When the insured life (x) lives longer than expected, the present value of the pension liabilities increases, because the pension plan will disburse payments under a long period. Mortality assumptions are the key parameters in measuring the longevity risk, which is the risk that annuitants live longer than expected, hence more fund is required to cover survival benefits. In order to measure the future annuity payments to present value, a discount rate is required. This rate essentially measures the expected rate of return on the pension fund's investments to reflect the rate of return assumed by the pension scheme's actuary. A higher discount rate tends to reduce the present value of liabilities because future payments will value less today and a lower discount rate will increase the present value of liabilities because the future payments will be more worth today. Consequently, the choice of discount rate markedly impacts the present value of future pension liabilities.

Life insurance and annuities benefits which are mortality rate dependent are the key life table functions governing life insurance operations to a large extent. In Koshkin and Gubina (2016) and Dmitriev and Kohkin (2018), life annuity was treated as statistical formulation without any evidence of applying survival probability function. There was no analytic evidence of interest rate intensity derivation necessary for life annuity function in the authors' work and it was not analytically expedient if the authors applied experienced based mortality. Furthermore, the authors did not apply parametric mortality modelling or polynomial derivation necessary for life annuity computations. Fahmy (2018) applied the gamma function to compare the analytic properties of the Makeham's law with extreme value distribution. Though the author's results showed that both Makeham's law and extreme value distribution have the same hazard function, the comparison lack explanations grounded in actuarial basis. It was not the author's focus to construct requisite life tables for the comparisons. Since the modal age at death x_M has more information across insured population in a more interpretable manner than the background mortality A, the modal age at death in the Makeham's law was not compared. Furthermore, basic life table quantities such the probability of death were not analytically derived.

Souza (2019) derived the upper and lower bounds for the fully continuous life annuity $\bar{a}_x$ and fully continuous life insurance $\bar{A}_x$ to address issues relating to incomplete life insurance data. However, the bounds derived were not tested under any known parsimonious mortality laws. Furthermore, the method of the integrated hazard function using known parametric or polynomial mortality intensities was not derived before the estimation of life annuity and life insurance benefits. Putra, Fitriyati and Mahmudi (2019) deployed the method of maximum likelihood estimation techniques to estimate the parameters of both Gompertz's parsimonious law of mortality function,

$$\mu_x = BC^x \tag{1}$$

and Makeham's parametric mortality law

$$\mu_x = A + BC^x \tag{2}$$

but the focus was not to construct the corresponding life tables. In both cases, the parameters estimated violated the permissible interval of validity $1.08 \le C \le 1.12$. Although, the log-likelihood function $\mathbf{L}(A,B,C)$ was derived through the Lagrange multipliers, a major limitation is that a singular Jacobian matrix J through first partial derivative equation of $\mathbf{L}$ with respect to parameters A, B, C was obtained which further constituted a numerical burden in estimating mortality trends. However, the authors acknowledged the inconsistencies in their method and could not offer convincing analytic

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explanations. Souza (2022) investigated the Makeham's mortality law using advanced algebraic method and stated closed form expressions for life annuity based on Makeham's law. It was not the author's interest to derive those close-form expressions and obtain the age where the survival function inflexes. Consequently, the mortality rate intensity was not obtained in terms of the modal age at death. Furthermore, the analytic form of the survival function in integral form based on the mortality law was not derived. The author did not seem to be aware that the mortality law satisfies the following property.

$$\lim_{\varepsilon \to 0}\left(\ell_{x-\varepsilon} - \ell_{x+\varepsilon}\right) = \lim_{x \to 0}\int_0^\infty \mu_{x+\xi}\left({}_{\xi}p_x\right)d\xi \tag{3}$$

It was not evident that the author addressed the shifting, compression and rectangularisation properties of the mortality function, the concepts describing the increased concentration of deaths at senescent ages such that the survival function ℓ_x forms a rectangular shape. The associated probability of death functions q_x and the number of deaths d_x at age x were not obtained. In Castellares, Patricio and Lemonte (2022), the statistical properties of Makeham's law were fully discussed as a distribution function rather than investigating the behaviour of the mortality intensity and its application in constructing a complete life table. As a result of this limitation, the estimation of the corresponding life table functions was not the interest. The Makeham's law assumes a background age independent mortality specific constant parameter which accounts for deaths resulting from accidental hump. Although, actuarial literatures adopted the law of mortality to produce mortality tables at various age ranges, however, the Makeham's law does not seem to generate a probability of death q_x exactly equal to 1.00 at some advanced ages. However, the model can produce probability of death high enough approaching a reasonable estimate of 1.00.

In Souza (2022), the hyper-geometric function was used to define the life expectancy of an insured at age x under the Makeham's law. However, the scope of the author's work did not cover the (i) use same hyper-geometric function to obtain the continuous whole life insurance benefit function, (ii) derivation of the basic death density formulation relating to Makeham's law and (iii) expression of the mortality death density in terms of indicator in the form

$$\mathbf{E}\left(I\left(\xi < T_x \le \xi + \Delta\xi\right)\right) = \frac{d}{d\xi}\left(-{}_{\xi}p_x\right) = \left({}_{\xi}p_x\mu_{x+\xi}\right) \tag{4}$$

to define death density. Castellares, Patricio and Lemonte (2022) essentially emphasized the increasing use of statistical methods of quantile function, characteristic function and moment generating function to model Makeham's distribution. Though the authors stated closed-form expression for the life expectancy e_x under Makeham's mortality function using the method of exponential integral of the type,

$$E_{\{R\}}\left(z\right) = z^{R-1}\Gamma\left(1-R, z\right) \tag{5}$$

for $R \in \mathbf{C}$ and $z \in \mathbf{C}$ and

$$E_{\{\xi\}}\left(z\right) = \int_1^\infty U^{-\xi}e^{-zU}dU \tag{6}$$

is the integro-exponential function for $s \in \mathbf{C}$ and $z \in \mathbf{C}$ such that $\mathrm{Re}\left(z\right) > 0$, the basic life table functions such as the survival function ℓ_x and ${}_{\xi}p_x$ were not derived. According to the authors' results, the life expectancy e_x of a life age x was alternatively expressed in the form of a gamma function but a degenerate form of life expectancy function under the Makeham's mortality intensity was obtained. Dragan (2022) obtained life expectancy e_x at age x for the Gamma-Gompertz using hypergeometric function

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$$ {}_2F_1\left(m,n,P;z\right)=\frac{\Gamma\left(P\right)}{\Gamma\left(n\right)\Gamma\left(P-n\right)}\int_0^1 U^{n-1}\left(1-U\right)^{P-n-1}\left(1-zu\right)^{-m}dU \tag{7} $$

and $|z|<1$; $\mathrm{Re}\left(P\right)>\mathrm{Re}\left(n\right)>0$, where $\Gamma(.)$ is the complete gamma function. This has partly solved the problem which Umar and Chukwudi (2019) could not provide. However, it was not the focus of the author to derive the integrated hazard function. It was observed that the author dwelled more on frailty analysis which does not have much bearing on complete life table construction.

Panjer and Tan (1995) constrained the Makeham's law to calibrate mortality rates for higher ages using the method of minimization. However, the authors did not attempt the derivation of key life table functions under algorithmic expansion but employed fitting. The mortality effect of intensity on mortality entropies was not justified analytically. Though the authors advanced evidence of a hump attributed to the excess of young adulthood which is a development of the additive constant in Makeham's second law representing accidental death independent of age, their interest did not focus on deriving (i) model for the modal age at death under this law (ii) the total severity in form of the integrated hazard function and (iii) the corresponding life insurance functions. Although the classical Makeham's law is useful for actuarial applications, its limitations highlight the importance of considering alternative models and additional factors when analysing mortality data. The classical Makeham's law of mortality which extends the Gompertz law by adding a constant term to account for factors like accidents or diseases that affect mortality, has several limitations. The model assumes that the relationship between age and mortality is consistent across all age groups, which may not hold true for younger populations. In pension scheme valuations, life annuity assumes an important role in estimating the present value of future pension liabilities. Actuaries numerically estimate the present value of these future annuity payments, which represents the cost of the pension benefits that the scheme owes to its participants.

Pension schemes may provide different types of annuities, and each has its own effect on the valuation. The estimation of the present value of liabilities using life annuities helps determine the funding status of the pension plan. If the present value of future pension obligations exceeds the scheme's assets, the plan is underfunded and may require additional contributions from the employer.

1.1 The Development of Mortality Function

$$ \mu_{x+\xi}=\frac{f_{T_x}\left(\xi\right)}{S_{T_x}\left(\xi\right)}=\frac{f_{T_x}\left(\xi\right)}{1-F_{T_x}\left(\xi\right)} \tag{8} $$

$$ \mu_{x+\xi}=\frac{f_{T_x}\left(\xi\right)}{1-\int_0^{\xi} f_{T_x}\left(\theta\right)d\theta} \tag{9} $$

$$ \int_0^{\xi}\mu_{x+\theta}d\theta=\int_0^{\xi}\left(\frac{f_{T_x}\left(\theta\right)}{1-\int_0^{\xi} f_{T_x}\left(\theta\right)d\theta}\right)d\theta=-\log_e\left(1-\int_0^{\xi} f_{T_x}\left(\theta\right)d\theta\right) \tag{10} $$

$$ \log_e\left(1-\int_0^{\xi} f_{T_x}\left(\theta\right)d\theta\right)=-\int_0^{\xi}\mu_{x+\theta}d\theta \tag{11} $$

$$ 1-\int_0^{\xi} f_{T_x}\left(\theta\right)d\theta=e^{-\int_0^{\xi}\mu_{x+\theta}d\theta} \tag{12} $$

From (8),

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$$S_{T_x}(\xi)=\frac{f_{T_x}(\xi)}{\mu_{x+\xi}} \tag{13}$$

$$f(\xi)=\mu_{x+\xi}\times e^{-\int_0^{\xi}\mu_{x+\theta}d\theta} \tag{14}$$

$$S_{T_x}(\xi)=\frac{\mu_{x+\xi}\times e^{-\int_0^{\xi}\mu_{x+\theta}d\theta}}{\mu_{x+\xi}}=e^{-\int_0^{\xi}\mu_{x+\theta}d\theta} \tag{15}$$

1.2 Generalised Life Insurance Integral Theorem

Given that the limits of summation of a mortality function $M(k)$ are $k=n$ and $k=N$ where

$$M(\xi)=e^{-\delta\xi}{}_{\xi}p_x\mu_{x+\xi} \tag{16}$$

then

$$\begin{aligned}\overline{A}^{1}_{x:\overline{m}|}=\int_0^m e^{-\delta\xi}{}_{\xi}p_x\mu_{x+\xi}d\xi=\sum_{k=0}^{m}M(k)-\frac{1}{2}M(0)-\frac{1}{2}M(m)-\sum_{k=2}^{\infty}\frac{B_k}{k!}M^{(k-1)}(m)\\+\sum_{k=2}^{\infty}\frac{B_k}{k!}M^{(k-1)}(0)\end{aligned} \tag{17}$$

Proof

There is no issue arising from defining the discrete term insurance $A^{1}_{x:\overline{m}|}$ purchased by (x) but the derivation of the continuous term insurance function $\overline{A}^{1}_{x:\overline{m}|}$ under the chosen mortality laws constitutes a challenge and hence a model is developed to address this gap. In the case of a finite sum, we have

$$\sum_{k=n}^{N}M(k)=\sum_{k=n}^{\infty}M(k)-\sum_{k=N+1}^{\infty}M(k)=\sum_{k=n}^{\infty}M(k)-\sum_{k=N}^{\infty}M(k)+M(N) \tag{18}$$

$$\sum_{k=n}^{N}M(k)=\sum_{k-n=0}^{\infty}M(k)-\sum_{k-N=0}^{\infty}M(k)+M(N) \tag{19}$$

Let $j=k-n\Rightarrow k=j+n$, then equation (19) becomes

$$\sum_{k=n}^{N}M(k)=\sum_{j=0}^{\infty}M(j+n)-\sum_{j=0}^{\infty}M(j+N)+M(N) \tag{20}$$

The equation (20) is the same as

$$\sum_{k=n}^{N}M(k)=\sum_{k=0}^{\infty}M(k+n)-\sum_{k=0}^{\infty}M(k+N)+M(N) \tag{21}$$

$$\begin{aligned}\sum_{k=n}^{N}M(k)=\int_n^{\infty}M(\alpha)d\alpha+\frac{1}{2}M(n)-\sum_{k=2}^{\infty}\frac{B_k}{k!}M^{(k-1)}(n)\\-\left(\int_N^{\infty}M(\alpha)d\alpha+\frac{1}{2}M(N)-\sum_{k=2}^{\infty}\frac{B_k}{k!}M^{(k-1)}(N)\right)+M(N)\end{aligned} \tag{22}$$

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$$\sum_{k=n}^{N} M(k)=\int_{n}^{\infty} M(\alpha)\, d\alpha+\frac{1}{2} M(n)-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(n)$$
$$-\int_{N}^{\infty} M(\alpha)\, d\alpha-\frac{1}{2} M(N)+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(N)+M(N) \tag{23}$$

$$\sum_{k=n}^{N} M(k)=\int_{n}^{\infty} M(\alpha)\, d\alpha-\int_{N}^{\infty} M(\alpha)\, d\alpha+\frac{1}{2} M(n)+\frac{1}{2} M(N)$$
$$+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(N)-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(n) \tag{24}$$

The first two integrals in equation (24) can be merged to have

$$\sum_{k=n}^{N} M(k)=\int_{n}^{N} M(\alpha)\, d\alpha+\frac{1}{2} M(n)+\frac{1}{2} M(N)+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(N)$$
$$-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(n) \tag{25}$$

Therefore, equation (25) can be rearranged to have

$$\int_{n}^{N} M(\alpha)\, d\alpha=\sum_{k=n}^{N} M(k)-\frac{1}{2} M(n)-\frac{1}{2} M(N)-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(N)$$
$$+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(n) \tag{26}$$

The model in (26) can hence be used to compute the continuous term insurance. Suppose $n=0$ and $N=m$ and $M(\xi)=e^{-\delta\xi}\,{}_{\xi}p_x\mu_{x+\xi}$ in equation (26) implies that

$$\bar{A}^{1}_{x:\overline{m}|}=\int_{0}^{m} e^{-\delta\xi}\,{}_{\xi}p_x\mu_{x+\xi}\, d\xi=\sum_{k=0}^{m} M(k)-\frac{1}{2} M(0)-\frac{1}{2} M(m)-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(m)$$
$$+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(0) \tag{27}$$

and letting $n=u$ and $N=u+m$ in (26), the continuous deferred insurance becomes

$${}_{u|}\bar{A}^{1}_{x:\overline{m}|}=\int_{u}^{u+m} e^{-\delta\xi}\,{}_{\xi}p_x\mu_{x+\xi}\, d\xi=\sum_{k=u}^{(u+m)} M(k)-\frac{1}{2} M(u)-\frac{1}{2} M(u+m)$$
$$-\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(u+m)+\sum_{k=2}^{\infty} \frac{B_k}{k!} M^{(k-1)}(u) \tag{28}$$

QED

It is not every individual who proposes to buy a life insurance product that can be accepted on normal premium rating due to bodily impairment, as a result of lifestyle, dysfunctional health conditions and environmental hazards. As a result, special policy terms can be computed on the basis of modified mortality table concerning extra risks but in practice this could be very difficult to obtain as only a rough approximation of the probable extra mortality risk may be available as result of shocks. If a life table is generated through a mortality law that predicts mortality rate which is lower than the

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actual mortality experienced by the insured, then the life insurers will run into losses. Suppose the mortality intensity $\mu_{x+\xi}$ is perturbed by a shock $\Delta_{x+\xi} > 0$.
where the intensity $\mu_{x+\xi}$ before shock increases to $\hat{\mu}_{x+\xi}$ after-shock. Then for any shock function $\Delta_{x+\xi} > 0$, the mortality intensity can be expressed as

$$\hat{\mu}_{x+\xi} = \mu_{x+\xi} + \Delta_{x+\xi} \tag{29}$$

The governing probability of survival function is derived as

$$_{\xi}\hat{p}_x = \exp\left[-\int_0^{\xi}\left(\mu_{x+s} + \Delta_{x+s}\right)ds\right] = \exp\left[-\int_0^{\xi}\mu_{x+s}ds\right]\exp\left[-\int_0^{\xi}\Delta_{x+s}ds\right] \tag{30}$$

$$_{\xi}\hat{p}_x = \left(_{\xi}p_x\right)\exp\left[-\int_0^{\xi}\Delta_{x+s}ds\right] < \left(_{\xi}p_x\right) \tag{31}$$

Thus, two inferences can be drawn from the result in (31): the survival probability after shock is less that survival probability before the shock $\Delta_{x+\xi}$ and that as the force of mortality increases, the survival probability function decreases till death. Since term insurance $A^1_{x:\overline{m|}}$ is an increasing function, this will further imply that the actuarial present value of death benefit $\hat{A}^1_{x:\overline{m|}}$ of 1 after-shock will be greater than the death benefit $A^1_{x:\overline{m|}}$ before shock for the discrete term insurance function in equation (32)

$$\hat{A}^1_{x:\overline{m|}} = \sum_{k=0}^{m-1} e^{-\delta k}\left(_{k|1}\hat{q}_x\right) > \sum_{k=0}^{m-1} e^{-\delta k}\left(_{k|1}q_x\right) = A^1_{x:\overline{m|}} \tag{32}$$

$$\left(_{k|1}q_x\right) = \left(\frac{\ell_{x+k}}{\ell_x} - \frac{\ell_{x+k+1}}{\ell_x}\right) = {_k}p_x - {_{k+1}}p_x \tag{33}$$

and m is the term in years. The 1 on top of x means that the insured aged x must die before the term of insurance m expires. Therefore, the amount of loss $\mathbf{L}_x$ to the life insurer after the shock is defined as

$$\mathbf{L}_x = \hat{A}^1_{x:\overline{m|}} - A^1_{x:\overline{m|}} > 0 \tag{34}$$

while the amount of profit $\mathbf{P}_x$ to the life insurer after the shock is defined as

$$\mathbf{P}_x = A_{x:\overset{1}{\overline{m|}}} - \hat{A}_{x:\overset{1}{\overline{m|}}} = v^m{_m}p_x - v^m{_m}\hat{p}_x \tag{35}$$

The 1 on top of m means that the term m of insurance must expire before the insured dies.

1.3 Benefit Function Theorem

Let $v_h = e^{-\delta h}$ denote the discount per subinterval

Define

$$_{m}p_x^{(h)} = {_{mh}}p_x = \mathbf{P}\left(T_x > mh\right)$$
$$q_{x+mh}^{(h)} = 1 - {_h}p_{x+mh} = \mathbf{P}\left(T_x \in \left(mh, (m+1)h\right]\right)$$

then

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(i) $\overline{A}_x = 1 - \delta \overline{a}_x = \mathbf{E}\left(e^{-\delta T_x}\right)$

(ii) $A_x^{(h)} = A_x + \lim_{h\to 0} O\left(h^2\right) \Rightarrow A_x^{(h)} - A_x = \lim_{h\to 0} O\left(h^2\right)$

(iii) $A_x^{(h)} = \overline{A}_x - \frac{h}{2} E\left(e^{-\delta T_x} \mu_{x+T_x}\right) + \lim_{h\to 0} O\left(h^3\right)$

Proof of (i)

Suppose the time axis is discretised into small equal intervals of width $h > 0$. and let grid points $\xi_m = mh$, $m = 0,1,2,3,...$ and let $v_h = e^{-\delta h}$ denote the discount per subinterval

$$_m p_x^{(h)} = _{mh} p_x = \mathbf{P}\left(T_x > mh\right) \tag{36}$$

$$q_{x+mh}^{(h)} = 1 - _h p_{x+mh} = \mathbf{P}\left(T_x \in \left(mh, (m+1)h\right]\right) \tag{37}$$

Then for each h, define the discretised life annuity and discretised life insurance as

$$a_x^{(h)} = h\sum_{k=0}^{\infty} v_h^m \times_m p_x^{(h)} \tag{38}$$

$$A_x^{(h)} = \sum_{m=0}^{\infty} v_h^{m+1} \times_m p_x^{(h)} q_{x+mh}^{(h)} \tag{39}$$

The factors of h make these Riemann sum approximations of the integrals defining $\overline{a}_x$ and $\overline{A}_x$
From the discrete-time actuarial identity, for the life aged x with step size h

$$A_x^{(h)} = 1 - d_h a_x^{(h)} \tag{40}$$

where the effective discount rate per step is

$$d_h = 1 - v_h = 1 - e^{-\delta h} \tag{41}$$

Substitute $a_x^{(h)}$ and $A_x^{(h)}$ into this

$$\sum_{m=0}^{\infty} v_h^{m+1} \times_m p_x^{(h)} q_{x+mh}^{(h)} = 1 - \left(1 - e^{-\delta h}\right) h \sum_{m=0}^{\infty} v_h^m \times_m p_x^{(h)} \tag{42}$$

This equation is exact for each, $h > 0$.

Now evaluate the limit as $h \to 0$ and show that it converges to $\overline{A}_x = 1 - \delta \overline{a}_x$

Define:

$$\phi_h = \sum_{m=0}^{\infty} v_h^{m+1} \times_m p_x^{(h)} q_{x+mh}^{(h)} \tag{43}$$

$$\gamma_h = h\sum_{m=0}^{\infty} v_h^m \times_m p_x^{(h)} \tag{44}$$

Then

$$\phi_h = 1 - \left(1 - e^{-\delta h}\right)\gamma_h \tag{45}$$

The intention is to evaluate the limit $h \to 0$.

$$\gamma_h = \sum_{m=0}^{\infty} h v_h^m \times_m p_x^{(h)} = \sum_{m=0}^{\infty} h e^{-\delta mh} \times_m p_x^{(h)} \tag{46}$$

As $h \to 0$, this defines the Riemann sum of the continuous integral

$$\lim_{h\to 0} \gamma_h = \lim_{h\to 0}\left(\sum_{m=0}^{\infty} h e^{-\delta mh} \times_m p_x^{(h)}\right) = \int_0^{\infty} e^{-\delta \xi}\left(_\xi p_x\right) d\xi = \overline{a}_x \tag{47}$$

hence $\gamma_h \to \overline{a}_x$

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$$q_{x+mh}^{(h)} = 1 - {}_h p_{x+mh} = 1 - e^{-\int_0^h \mu_{x+kh+u} du} \tag{48}$$

as $h \to 0$, apply the first-order expansion $1 - e^{-X} = X + O\left(X^2\right)$ yields

$$q_{x+mh}^{(h)} = h\mu_{x+mh} + O\left(h^2\right) \tag{49}$$

hence

$$_m p_x^{(h)} q_{x+mh}^{(h)} = {}_{mh} p_x \left(h\mu_{x+mh} + O\left(h^2\right)\right) \tag{50}$$

Substitute this into ϕ_h

$$\phi_h = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times {}_{mh} p_x \left(h\mu_{x+mh} + O\left(h^2\right)\right) \tag{51}$$

Simplify,

$$\begin{aligned}\phi_h &= \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times {}_{mh} p_x \left(h\mu_{x+mh}\right) + \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times {}_{mh} p_x \left(O\left(h^2\right)\right) \\ &= \sum_{m=0}^{\infty} h e^{-\delta t_m} \times \left({}_{\xi} p_x\right) \mu_{x+\xi_m} + O\left(h^2\right)\end{aligned} \tag{52}$$

As $h \to 0$, this Riemann sum converges to

$$\lim_{h\to 0} \phi_h = \lim_{h\to 0} \left(\sum_{m=0}^{\infty} e^{-\delta t_m} \times {}_{\xi} p_x h\mu_{x+\xi_m} + O(h)\right) = \int_0^{\infty} e^{-\delta\xi} \left({}_{\xi} p_x\right) \mu_{x+\xi} d\xi = \overline{A}_x \tag{53}$$

Hence $\phi_h \to \overline{A}_x$

$$d_h = 1 - e^{-\delta h} = \delta h - \frac{\delta^2 h^2}{2} + O\left(h^3\right) \tag{54}$$

$$\delta = \lim_{h\to 0} \frac{d_h}{h} = \lim_{h\to 0} \left(\frac{\delta h - \frac{\delta^2 h^2}{2} + O\left(h^3\right)}{h}\right) \tag{55}$$

Divide both sides of $\phi_h = 1 - d_h \gamma_h$ by 1 (without scaling), and take limits as $h \to 0$

$$\lim_{h\to 0} \phi_h = \lim_{h\to 0} \left(1 - d_h \gamma_h\right) \Rightarrow \overline{A}_x = 1 - \lim_{h\to 0} d_h \gamma_h \tag{56}$$

But since

$$\lim_{h\to 0} d_h \gamma_h = \lim_{h\to 0} \left(\delta h + O\left(h^2\right)\right) \frac{\gamma_h}{h} = \delta \lim_{h\to 0} \gamma_h = \delta \overline{a}_x \tag{56a}$$

Consequently,

$$\overline{A}_x = 1 - \delta \overline{a}_x \tag{56b}$$

Proof of (ii)

From the expansions,

$$q_{x+mh}^{(h)} = h\mu_{x+mh} + O\left(h^2\right) \tag{56c}$$

$$d_h = 1 - e^{-\delta h} = \delta h - \frac{\delta^2 h^2}{2} + O\left(h^3\right) \tag{56d}$$

$$A_x^{(h)} = \sum_{m=0}^{\infty} v_h^{m+1} \times {}_m p_x^{(h)} q_{x+mh}^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times {}_m p_x^{(h)} q_{x+mh}^{(h)} \tag{56e}$$

Substituting (49) into (56e), the discrete insurance definition yields

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$$A_x^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(h\mu_{x+mh} + O\left(h^2\right) \right) \tag{56f}$$

$$A_x^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(h\mu_{x+mh} \right) + \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(O\left(h^2\right) \right) \tag{56g}$$

$$A_x^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(h\mu_{x+mh} \right) + O\left(h^2\right) \tag{56h}$$

$$A_x^{(h)} = h\sum_{m=0}^{\infty} e^{-\delta\xi_m} \left({}_{\xi} p_x \right)\mu_{x+\xi_m} + O\left(h^2\right) \tag{56i}$$

hence, as $h \to 0$

$$\lim_{h\to 0} A_x^{(h)} = \lim_{h\to 0}\left[h\sum_{m=0}^{\infty} e^{-\delta\xi_m} \left({}_{\xi} p_x \right)\mu_{x+\xi_m} \right] + \lim_{h\to 0} O\left(h^2\right) \tag{56j}$$

$$A_x^{(h)} = A_x + \lim_{h\to 0} O\left(h^2\right) \Rightarrow A_x^{(h)} - A_x = \lim_{h\to 0} O\left(h^2\right) \tag{56k}$$

Hence this defines first-order convergence from the discrete case to the continuous case. Therefore, the numerical evaluation of continuous life insurance function applying small discrete steps has an error of order $\lim_{h\to 0} O\left(h^2\right)$ for actuarial valuation.

Proof of (iii)

Furthermore, for the second-order discretization correction, recall in equation (40) that

$$A_x^{(h)} = 1 - d_h a_x^{(h)} \tag{56l}$$

where the effective discount rate per step is $d_h = 1 - v_h = 1 - e^{-\delta h}$. Suppose d_h and the mortality term $q_{x+mh}^{(h)}$ are expanded to second order

$$d_h = 1 - e^{-\delta h} = \delta h - \frac{\delta^2 h^2}{2} + O\left(h^3\right) \tag{56m}$$

$$q_{x+mh}^{(h)} = 1 - e^{-\int_0^h \mu_{x+mh+s} ds} = h\mu_{x+mh} - \frac{1}{2}h^2\mu_{x+mh}^2 + O\left(h^3\right) \tag{56n}$$

Substituting (56n) into (56o), the discrete insurance definition yields

$$A_x^{(h)} = \sum_{m=0}^{\infty} v_h^{m+1} \times_m p_x^{(h)} q_{x+mh}^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} q_{x+mh}^{(h)} \tag{56o}$$

$$A_x^{(h)} = \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(h\mu_{x+mh} - \frac{1}{2}h^2\mu_{x+mh}^2 + O\left(h^3\right) \right) \tag{56p}$$

$$\begin{aligned} A_x^{(h)} = {} & \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} h\mu_{x+mh} - \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(-\frac{1}{2}h^2\mu_{x+mh}^2 \right) \\ & + \sum_{m=0}^{\infty} e^{-\delta(m+1)h} \times_m p_x^{(h)} \left(O\left(h^3\right) \right) \end{aligned} \tag{56q}$$

$$\begin{aligned} A_x^{(h)} = {} & h\sum_{m=0}^{\infty} e^{-\delta t_m} \left({}_{\xi} p_x \right)\mu_{x+\xi_m} - \frac{1}{2}h^2 \sum_{m=0}^{\infty} e^{-\delta\xi_m} \times \left({}_{\xi} p_x \right)\left(\mu_{x+\xi_m}^2 \right) \\ & + O\left(h^3\right) \end{aligned} \tag{56r}$$

hence, as $h \to 0$

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$$\left(A_x^{(h)}\right)=\sum_{m=0}^{\infty}he^{-\delta\xi_m}\left({}_{\xi}p_x\right)\mu_{x+\xi_m}-\frac{1}{2}h\sum_{m=0}^{\infty}he^{-\delta\xi_m}\times\left({}_{\xi}p_x\right)\left(\mu^2_{x+\xi_m}\right)+O\left(h^3\right) \quad (56s)$$

$$\lim_{h\to 0}\left(A_x^{(h)}\right)=\lim_{h\to 0}\left(\sum_{m=0}^{\infty}e^{-\delta\xi_m}\left({}_{\xi}p_x\right)\mu_{x+\xi_m}\right)-\frac{h}{2}\lim_{h\to 0}\left[h\sum_{m=0}^{\infty}e^{-\delta\xi_m}\times\left({}_{\xi}p_x\right)\left(\mu^2_{x+\xi_m}\right)\right]+\lim_{h\to 0}O\left(h^3\right) \quad (56t)$$

$$A_x^{(h)}=\overline{A}_x-\frac{h}{2}\int_0^{\infty}e^{-\delta\xi}\mu^2_{x+\xi}\left({}_{\xi}p_x\right)d\xi+\lim_{h\to 0}O\left(h^3\right) \quad (56u)$$

$$A_x^{(h)}=\overline{A}_x-\frac{h}{2}E\left(e^{-\delta T_x}\mu_{x+T_x}\right)+\lim_{h\to 0}O\left(h^3\right) \quad (56w)$$

1.3 Method of Deriving the Severity to Die Function

Consider the modified Makeham's law defined in (57). Consider the modified Makeham's mortality function (57) where A, B, C, H are mortality parameters satisfying Occam's razor and α is a parameter governing the curvature of the trajectory of (57).

$$\mu_x = A + Hx^{\alpha} + BC^x, \quad \alpha\in R^+ \quad (57)$$

To obtain the integrated hazard function, integrate both sides of (57) from zero to an arbitrary age x of the insured to obtain the severity to die function as

$$\int_0^x \mu_{\xi}d\xi=\int_0^x\left(A+H\xi^{\alpha}+BC^{\xi}\right)d\xi \quad (58)$$

$$\Rightarrow\int_0^x \mu_{\xi}d\xi=\left[A\xi+\frac{Ht^{(1+\alpha)}}{(1+\alpha)}+\frac{BC^{\xi}}{\log_e C}\right]_0^x \quad (59)$$

$$\Rightarrow\int_0^x \mu_{\xi}d\xi=Ax+\frac{Hx^{(1+\alpha)}}{(1+\alpha)}+\frac{BC^x}{\log_e C}-\frac{B}{\log_e C} \quad (60)$$

In order to simplify (60), we apply

$$\begin{cases} A=\log_e S^{-1}=(-1)\log_e S \\ H=\log_e W^{-(1+\alpha)}=-(1+\alpha)\times\log_e W \\ B=-\left(\log_e C\right)\left(\log_e g\right) \\ \log_e g=\dfrac{-B}{\log_e C} \\ K=\dfrac{\ell_0}{g} \end{cases} \quad (61)$$

Substitute the transformations in (61) into (60) to obtain

$$\Rightarrow\int_0^x \mu_{\xi}d\xi=(-1)\times x\times\log_e S-\frac{(1+\alpha)\times\log_e W\times x^{(1+\alpha)}}{(1+\alpha)}-\frac{\left(\log_e C\right)\left(\log_e g\right)\times C^x}{\log_e C}-\frac{-\left(\log_e C\right)\left(\log_e g\right)}{\log_e C} \quad (62)$$

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$$\int_0^x \mu_\xi d\xi = (-1)\times x \times \log_e S - \log_e W \times x^{(1+\alpha)} - (\log_e g)\times C^x + (\log_e g) \tag{63}$$

$$\int_0^x \mu_\xi d\xi = -(\log_e S)x - (\log_e W)x^{(1+\alpha)} - (\log_e g)C^x + (\log_e g) \tag{64}$$

$$\Rightarrow \int_0^x \mu_\xi d\xi = -x\log_e S - x^{(1+\alpha)}\log_e W - C^x \log_e g + \log_e g \tag{65}$$

$$\Rightarrow \int_0^x \mu_\xi d\xi = -\log_e S^x - \log_e W^{x^{(1+\alpha)}} - \log_e g^{C^x} + \log_e g \tag{66}$$

The survival probability function is given by

$$ {}_\xi p_x = \frac{\ell_{x+\xi}}{\ell_x} = \exp\left(-\int_0^\xi \mu_{x+\theta} d\theta\right) \tag{67}$$

$$ {}_\xi p_0 = \exp\left(-\int_0^\xi \mu_\theta d\theta\right) \tag{68}$$

$$\ell_\xi = \ell_0 \exp\left(-\int_0^\xi \mu_\theta d\theta\right) \tag{69}$$

$$\ell_x = \ell_0 \exp\left(-\int_0^x \mu_\xi d\xi\right) \tag{70}$$

Substitute (66) into (70) yields

$$\ell_x = \ell_0 e^{-\left[-\log_e S^x - \log_e W^{x^{(1+\alpha)}} - \log_e g^{C^x} + \log_e g\right]} = l_0 e^{\left[\log_e S^x + \log_e W^{x^{(1+\alpha)}} + \log_e g^{C^x} - \log_e g\right]} \tag{71}$$

Simplify (71) to get

$$\ell_x = \ell_0 e^{\left[\log_e\left(\frac{s^x W^{x^{(1+\alpha)}} g^{C^x}}{g}\right)\right]} = \ell_0 \frac{s^x W^{x^{(1+\alpha)}} g^{C^x}}{g} \tag{72}$$

The number of lives surviving to age x is given as

$$\ell_x = \frac{\ell_0}{g} S^x W^{x^{(1+\alpha)}} g^{C^x} = KS^x W^{x^{(1+\alpha)}} g^{C^x}; \quad K = \frac{\ell_0}{g} \tag{73}$$

Replace x by $x+\xi$ in (73) to get

$$\ell_{x+\xi} = KS^{(x+\xi)} W^{(x+\xi)^{(1+\alpha)}} g^{C^{(x+\xi)}} \tag{74}$$

The probability that a life aged x survives to age $x+\xi$ is obtained by dividing (74) by (73):

$$ {}_\xi p_x = \frac{KS^{(x+\xi)} W^{(x+t)^{(1+\alpha)}} g^{C^{(x+\xi)}}}{KS^x W^{x^{(1+\alpha)}} g^{C^x}} = S^t W^{(x+\xi)^{(1+\alpha)} - x^{(1+\alpha)}} g^{C^{(x+\xi)} - C^x} \tag{75}$$

Substitute since α is real, we choose $\alpha = 1$ to yield

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$$ {}_{\xi}p_{x} = S^{\xi} W^{\left(2x\xi+\xi^{2}\right)} g^{C^{(x+\xi)}-C^{x}} \tag{76}$$

1.4 A Novel Method of Deriving the Bernoulli Coefficients

Define the operator T as

$$T = D + \frac{D^2}{2!} + \frac{D^3}{3!} + \frac{D^4}{4!} + \frac{D^5}{5!} + \frac{D^6}{6!} + \ldots = \sum_{m=1}^{\infty} \frac{D^m}{m!} \tag{87}$$

$$T = \sum_{m=1}^{\infty} \frac{D^m}{m!} - 1 = e^D - 1 \tag{88}$$

We hence seek the inverse operator T^{-1}

$$T^{-1} = \left(\sum_{m=1}^{\infty} \frac{D^m}{m!} \right)^{-1} \tag{89}$$

The inverse operator T^{-1} acts on the continuous death density function f as $T^{-1}\left[f\right]\left(x\right)$

$$e^D f\left(x\right) = f\left(x+1\right) \tag{90}$$

$$\left(e^D - 1\right) f\left(x\right) = e^D f\left(x\right) - f\left(x\right) \tag{91}$$

$$\left(e^D - 1\right) f\left(x\right) = f\left(x+1\right) - f\left(x\right) \tag{92}$$

$$T = \Delta = e^D - 1 = \Delta f\left(x\right) = f\left(x+1\right) - f\left(x\right) \tag{93}$$

$$T^{-1}\left[f\right]\left(x\right) = \left(e^D - 1\right)^{-1} f\left(x\right) \tag{94}$$

For any function $f\left(x\right)$, we seek a function $h\left(x\right)$ such that

$$\Delta h\left(x\right) = h\left(x+1\right) - h\left(x\right) = f\left(x\right) \Rightarrow h\left(x+1\right) = h\left(x\right) + f\left(x\right) \tag{95}$$

This is a recurrence relation and we can solve it step by step.
Fix the initial condition

$$h\left(\alpha\right) = \beta \tag{96}$$

$$h\left(\alpha+1\right) = h\left(\alpha\right) + f\left(\alpha\right) = \beta + f\left(\alpha\right) \tag{97}$$

$$h\left(\alpha+2\right) = h\left(\alpha+1\right) + f\left(\alpha+1\right) = \beta + f\left(\alpha\right) + f\left(\alpha+1\right) \tag{98}$$

$$h\left(\alpha+3\right) = h\left(\alpha+2\right) + f\left(\alpha+2\right) = \beta + f\left(\alpha\right) + f\left(\alpha+1\right) + f\left(\alpha+2\right) \tag{99}$$

$$h\left(\alpha+4\right) = h\left(\alpha+3\right) + f\left(\alpha+3\right) = \beta + f\left(\alpha\right) + f\left(\alpha+1\right) + f\left(\alpha+2\right) + f\left(\alpha+3\right) \tag{100}$$

$$h\left(\alpha+m\right) = \alpha + \sum_{r=0}^{m-1} f\left(\alpha+r\right) \tag{101}$$

Set $\alpha = 0$, then

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$$h(m)=\sum_{r=0}^{m-1} f(r)=f(0)+f(1)+f(2)+f(3)+...+f(m-1) \tag{102}$$

Therefore,
$h(x)=T^{-1}$ is the discrete antiderivative representing the cumulative sum. Now

$$T=\lambda(D)=D+\frac{D^2}{2!}+\frac{D^3}{3!}+\frac{D^4}{4!}+\frac{D^5}{5!}+\frac{D^6}{6!}+...=D\left(1+\frac{D}{2!}+\frac{D^2}{3!}+\frac{D^3}{4!}+\frac{D^4}{5!}+\frac{D^5}{6!}+...\right) \tag{103}$$

$$\lambda(D)=1+\frac{D}{2!}+\frac{D^2}{3!}+\frac{D^3}{4!}+\frac{D^4}{5!}+\frac{D^5}{6!}+...=\sum_{m=0}^{\infty}\frac{D^m}{(m+1)!}\Rightarrow T=D\lambda(D) \tag{104}$$

$$T^{-1}=\left(\lambda(D)\right)^{-1}D^{-1} \tag{105}$$

$$\lambda(D)=1+\frac{D}{2!}+\frac{D^2}{3!}+\frac{D^3}{4!}+\frac{D^4}{5!}+\frac{D^5}{6!}+... \tag{106}$$

Expand $T^{-1}=\left(\lambda(D)\right)^{-1}$ in power series of the form

$$\left(\lambda(D)\right)^{-1}=b_0+b_1D+b_2D^2+b_3D^3+b_4D^4+... \tag{107}$$

Obviously,

$$\lambda(D)\times\left(\lambda(D)\right)^{-1}=1 \tag{108}$$

$$b_0=1 \tag{109}$$

We create the identity

$$\sum_{k=0}^{m}\frac{1}{(k+1)!}b_{m-k}\equiv 0 \tag{110}$$

where m represents the position of the coefficients to be calculated.
Set $m=1$

$$\sum_{k=0}^{1}\frac{1}{(k+1)!}b_{1-k}\equiv 0 \tag{111}$$

$$\sum_{k=0}^{1}\frac{1}{(k+1)!}b_{1-k}=\frac{1}{(0+1)!}b_{1-0}+\frac{1}{(1+1)!}b_{1-1}=0 \tag{112}$$

$$\sum_{k=0}^{1}\frac{1}{(k+1)!}b_{1-k}=b_1+\frac{1}{2!}b_0=0 \tag{113}$$

$$\sum_{k=0}^{1}\frac{1}{(k+1)!}b_{1-k}=b_1+\frac{1}{2!}=0 \tag{114}$$

$$b_1=-\frac{1}{2} \tag{115}$$

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Set $m = 2$

$$\sum_{k=0}^{2}\frac{1}{(k+1)!}b_{2-k} = 0 \tag{116}$$

$$\sum_{k=0}^{2}\frac{1}{(k+1)!}b_{2-k} = \frac{1}{(0+1)!}b_{2-0} + \frac{1}{(1+1)!}b_{2-1} + \frac{1}{(2+1)!}b_{2-2} = 0 \tag{117}$$

$$\sum_{k=0}^{2}\frac{1}{(k+1)!}b_{2-k} = \frac{1}{1!}b_2 + \frac{1}{2!}b_1 + \frac{1}{3!}b_0 = 0 \tag{118}$$

$$\sum_{k=0}^{2}\frac{1}{(k+1)!}b_{2-k} = \frac{1}{1}b_2 + \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{6}\times(1) = 0 \tag{119}$$

$$\sum_{k=0}^{2}\frac{1}{(k+1)!}b_{2-k} = b_2 + \frac{1}{6} - \frac{1}{4} = 0 \Rightarrow b_2 + \frac{2-3}{12} = 0 \Rightarrow b_2 = \frac{1}{12} \tag{120}$$

Set $m = 3$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} \equiv 0 \tag{121}$$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} = \frac{1}{(0+1)!}b_{3-0} + \frac{1}{(1+1)!}b_{3-1} + \frac{1}{(2+1)!}b_{3-2} + \frac{1}{(3+1)!}b_{3-3} = 0 \tag{122}$$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} = \frac{1}{1!}b_3 + \frac{1}{2!}b_2 + \frac{1}{3!}b_1 + \frac{1}{4!}b_0 = 0 \tag{123}$$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} = b_3 + \frac{1}{2}\left(\frac{1}{12}\right) + \frac{1}{6}\left(-\frac{1}{2}\right) + \frac{1}{24}(1) = 0 \tag{124}$$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} = b_3 + \frac{1}{24} - \frac{1}{12} + \frac{1}{24} = 0 \tag{125}$$

$$\sum_{k=0}^{3}\frac{1}{(k+1)!}b_{3-k} = b_3 + \frac{2}{24} - \frac{1}{12} = b_3 + \frac{1}{12} - \frac{1}{12} = 0 \Rightarrow b_3 = 0 \tag{126}$$

Set $m = 4$

$$\sum_{k=0}^{4}\frac{1}{(k+1)!}b_{4-k} = \frac{1}{(0+1)!}b_{4-0} + \frac{1}{(1+1)!}b_{4-1} + \frac{1}{(2+1)!}b_{4-2} + \frac{1}{(3+1)!}b_{4-3} + \frac{1}{(4+1)!}b_{4-4} = 0 \tag{127}$$

$$\sum_{k=0}^{4}\frac{1}{(k+1)!}b_{4-k} = \frac{1}{1!}b_{4-0} + \frac{1}{2!}b_3 + \frac{1}{3!}b_2 + \frac{1}{4!}b_1 + \frac{1}{5!}b_0 = 0 \tag{128}$$

$$\sum_{k=0}^{4}\frac{1}{(k+1)!}b_{4-k} = b_4 + \frac{1}{2}(0) + \frac{1}{6}\left(\frac{1}{12}\right) + \frac{1}{24}\left(-\frac{1}{2}\right) + \frac{1}{120}(1) = 0 \tag{129}$$

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$$\sum_{k=0}^{4}\frac{1}{(k+1)!}b_{4-k}=b_4+\frac{1}{72}-\frac{1}{48}+\frac{1}{120}=0 \tag{130}$$

$$\sum_{k=0}^{4}\frac{1}{(k+1)!}b_{4-k}=b_4+\frac{1}{72}-\frac{1}{48}+\frac{1}{120}=0\Rightarrow b_4+\frac{1}{720}=0\Rightarrow b_4=-\frac{1}{720} \tag{131}$$

Set $m=5$

$$\sum_{k=0}^{5}\frac{1}{(k+1)!}b_{5-k}=\frac{1}{(0+1)!}b_{5-0}+\frac{1}{(1+1)!}b_{5-1}+\frac{1}{(2+1)!}b_{5-2}+\frac{1}{(3+1)!}b_{5-3}+\frac{1}{(4+1)!}b_{5-4}+\frac{1}{(5+1)!}b_{5-5}=0 \tag{132}$$

$$\sum_{k=0}^{5}\frac{1}{(k+1)!}b_{5-k}=\frac{1}{1!}b_5+\frac{1}{2!}b_4+\frac{1}{3!}b_3+\frac{1}{4!}b_2+\frac{1}{5!}b_1+\frac{1}{6!}b_0=0 \tag{133}$$

$$\sum_{k=0}^{5}\frac{1}{(k+1)!}b_{5-k}=b_5+\frac{1}{2}\left(-\frac{1}{720}\right)+\frac{1}{6}(0)+\frac{1}{24}\left(\frac{1}{12}\right)+\frac{1}{120}\left(-\frac{1}{2}\right)+\frac{1}{720}(1)=0 \tag{134}$$

$$\sum_{k=0}^{5}\frac{1}{(k+1)!}b_{5-k}=b_5-\frac{1}{1440}+\frac{1}{288}-\frac{1}{240}+\frac{1}{720}=0 \tag{135}$$

$$\sum_{k=0}^{5}\frac{1}{(k+1)!}b_{5-k}=b_5+\frac{1}{720}+\frac{1}{288}-\frac{1}{1440}-\frac{1}{240}=0\Rightarrow b_5+0=0\Rightarrow b_5=0 \tag{136}$$

2.0 Material and Methods

The Continuous whole life annuity

$$\bar{a}_x=\int_0^{\infty}e^{-\delta\xi}\left({}_{\xi}p_x\right)d\xi \tag{137}$$

$${}_{\xi}p_x=s^{\xi}W^{2x\xi+\xi^2}g^{C^{x+\xi}-C^x} \tag{138}$$

$$\bar{a}_x=\int_0^{\infty}e^{-\delta\xi}s^{\xi}W^{2x\xi+\xi^2}g^{C^{x+\xi}-C^x}d\xi \tag{139}$$

The function in equation (137) under the Generalized Makeham's law is analytically intractable. To overcome the problem, the Woolhouse series formula is deployed. In (137), define the function

$$f(\xi)=e^{-\delta\xi}\left({}_{\xi}p_y\right) \tag{140}$$

$$\mu_y=A+Hy+BC^y \tag{141}$$

$$\mu'_y=H+B\left(\log_e C\right)C^y \tag{142}$$

$$\mu''_y=B\left(\log_e C\right)^2C^y \tag{143}$$

$$\lim_{\xi\to\infty}f(\xi)=\lim_{\xi\to\infty}\left\{e^{-\delta\xi}\left({}_{\xi}p_y\right)\right\}=0 \tag{144}$$

Using (140) yields

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$$f(0)=1 \tag{145}$$

$$f^{(1)}(0)=-f(0)\left\{\delta+\mu_y\right\}=-\left\{\delta+\mu_y\right\} \tag{146}$$

$$\Rightarrow f^{(2)}(0)=\left(\delta+\mu_y\right)^2-\left\{\mu_y'\right\} \tag{147}$$

$$\Rightarrow f^{(3)}(0)=2\left(\delta+\mu_y\right)\mu_y'-\left(\delta+\mu_y\right)^3+\mu_y'\left(\delta+\mu_y\right)-\mu_y'' \tag{148}$$

Let Δ be the proper fraction of the insurance period, then using the derived Bernoulli coefficients, we have

$$\int_0^\infty f(\xi)d\xi=\Delta\sum_{j=0}^\infty f(j\Delta)-\frac{\Delta}{2}f(0)+\frac{\Delta^2}{12}f^{(1)}(0)-\frac{\Delta^4}{720}f^{(3)}(0)+\ldots \tag{149}$$

where $\lim_{\xi\to\infty} f(\xi)=0$

Setting $\Delta=1$ in (149) yields

$$\int_0^\infty f(\xi)d\xi\cong\sum_{j=0}^\infty f(j)-\frac{1}{2}f(0)+\frac{1}{12}f^{(1)}(0)-\frac{1}{720}f^{(3)}(0) \tag{150}$$

$$\begin{aligned}\Rightarrow\int_0^\infty f(\xi)d\xi&=\sum_{j=0}^\infty e^{-\delta j}\left({}_jP_y\right)-\frac{1}{2}-\frac{1}{12}\left\{\delta+\mu_y\right\}\\&-\frac{1}{720}\left[2\left(\delta+\mu_y\right)\mu_y'-\left(\delta+\mu_y\right)^3+\mu_y'\left(\delta+\mu_y\right)-\mu_y''\right]\end{aligned} \tag{151}$$

Using (138) and applying the first and second derivatives of the force of mortality μ_x, (151) simplifies to

$$\begin{aligned}\int_0^\infty f(\xi)d\xi&=\sum_{j=0}^\infty e^{-\delta j}\left(S^jW^{j^2+2yj}g^{C^{y+j}-C^y}\right)-\frac{1}{2}-\frac{1}{12}\left\{\delta+A+Hy+BC^y\right\}\\&-\frac{1}{720}\left\{\begin{array}{l}2\left(\delta+A+Hy+BC^y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+A+Hy+BC^y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+A+Hy+BC^y\right\}-\left\{\left(\log_e C\right)^2BC^y\right\}\end{array}\right\}\end{aligned} \tag{152}$$

Setting $\Delta=\frac{1}{k}$ in (149) and again using (138) and the first and second derivatives of the force of mortality μ_x (152) simplifies to

$$\int_0^\infty f(\xi)d\xi=\frac{1}{k}\sum_{j=0}^\infty f\left(\frac{j}{k}\right)-\frac{1}{2k}f(0)+\frac{1}{12k^2}f^{(1)}(0)-\frac{1}{720k^4}f^{(3)}(0) \tag{153}$$

Using (136)

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$$\Rightarrow \int_0^{\infty} f(\xi) d\xi = \frac{1}{k}\sum_{j=0}^{\infty} e^{-\frac{\delta j}{k}}\left(S^{\frac{j}{k}}W^{\frac{j^2}{k^2}+\frac{2yj}{k}}g^{C^{y+\frac{j}{k}}-C^y}\right)-\frac{1}{2k}-\frac{1}{12k^2}\left\{\delta+\mu_y\right\}$$
$$-\frac{1}{720k^4}\left\{\begin{array}{l}2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\} \quad (154)$$

The equations (152) and (154) represent same integral and can be set equal to each other

$$\frac{1}{k}\sum_{j=0}^{\infty} e^{-\frac{\delta j}{k}}\left(S^{\frac{j}{k}}W^{\frac{j^2}{k^2}+\frac{2yj}{k}}g^{C^{y+\frac{j}{k}}-C^y}\right)-\frac{1}{2k}-\frac{1}{12k^2}\left\{\delta+\mu_y\right\}$$
$$-\frac{1}{720k^4}\left\{\begin{array}{l}2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\}$$
$$=\sum_{j=0}^{\infty} e^{-\delta j}\left(S^{j}W^{j^2+2yj}g^{C^{y+j}-C^y}\right)-\frac{1}{2}-\frac{1}{12}\left\{\delta+\mu_y\right\} \quad (155)$$
$$-\frac{1}{720}\left\{\begin{array}{l}2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\}$$

$$\Rightarrow \frac{1}{k}\sum_{j=0}^{\infty} e^{-\frac{\delta j}{k}}\left(S^{\frac{j}{k}}W^{\frac{j^2}{k^2}+\frac{2yj}{k}}g^{C^{y+\frac{j}{k}}-C^y}\right)$$
$$=\sum_{j=0}^{\infty} e^{-\delta j}\left(S^{j}W^{j^2+2yj}g^{C^{y+j}-C^y}\right)$$
$$+\frac{1}{720k^4}\left\{\begin{array}{l}2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\} \quad (156)$$
$$-\frac{1}{720}\left\{\begin{array}{l}2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3\\+\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\}$$
$$+\frac{1}{2}\left(\frac{1}{k}-1\right)+\frac{1}{12}\left\{\delta+\mu_y\right\}\left(\frac{1}{k^2}-1\right)$$

The first term in (156) can be expressed as the LHS of (157) to describe life annuity payable m times to the annuitant starting from the retirement age y.

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$$\ddot{a}_y^{(k)} = \sum_{j=0}^{\Omega-y-1} e^{-\delta j}\left(S^j W^{j^2+2yj} g^{C^{y+j}-C^y}\right)$$
$$+\frac{1}{720k^4}\left\{\begin{array}{l} 2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3 \\ +\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\}$$
$$-\frac{1}{720}\left\{\begin{array}{l} 2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3 \\ +\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\}\end{array}\right\} \quad (157)$$
$$+\frac{1}{2}\left(\frac{1}{k}-1\right)+\frac{1}{12}\left\{\delta+\mu_y\right\}\left(\frac{1}{k^2}-1\right)$$

Following methodologies in Ogungbenle**,** Sirisena, Ukwu. and Adeyele (2025a), the force of interest δ in (157) is obtained from the equation

$$5\delta^4 - 300\delta^2 + 73800\delta - 3600 = 0 \quad (158)$$

3.0 Presentation of Results

TABLE 1: $GM(2,2)$ FEMALE LIFE INSURANCE AND ANNUITY TABLE

x	$\ddot{a}_x$	A_x	$\bar{a}_x$	$\bar{A}_x$	$P(\bar{A}_x)$
0	20.837337	0.007746	20.333263	0.031749	0.001524
1	20.833771	0.007916	20.329696	0.031919	0.001532
2	20.830151	0.008088	20.326076	0.032092	0.001541
3	20.826473	0.008263	20.322397	0.032267	0.001549
4	20.822730	0.008441	20.318653	0.032445	0.001558
5	20.818917	0.008623	20.314839	0.032627	0.001567
6	20.815026	0.008808	20.310948	0.032812	0.001576
7	20.811050	0.008998	20.306971	0.033001	0.001586
8	20.806980	0.009191	20.302901	0.033195	0.001595
9	20.802810	0.009390	20.298730	0.033394	0.001605
10	20.798528	0.009594	20.294447	0.033598	0.001615
11	20.794125	0.009804	20.290044	0.033807	0.001626
12	20.789592	0.010019	20.285510	0.034023	0.001637
13	20.784916	0.010242	20.280833	0.034246	0.001648
14	20.780086	0.010472	20.276003	0.034476	0.001659
15	20.775090	0.010710	20.271006	0.034714	0.001671
16	20.769915	0.010956	20.265831	0.034960	0.001683
17	20.764548	0.011212	20.260463	0.035216	0.001696
18	20.758976	0.011477	20.254890	0.035481	0.001709
19	20.753184	0.011753	20.249098	0.035757	0.001723
20	20.747160	0.012040	20.243074	0.036044	0.001737
21	20.740890	0.012339	20.236802	0.036343	0.001752
22	20.734358	0.012650	20.230271	0.036654	0.001768

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23	20.727554	0.012974	20.223465	0.036978	0.001784
24	20.720463	0.013311	20.216374	0.037316	0.001801
25	20.713073	0.013663	20.208983	0.037667	0.001819
26	20.705373	0.014030	20.201282	0.038034	0.001837
27	20.697350	0.014412	20.193259	0.038416	0.001856
28	20.688996	0.014810	20.184905	0.038814	0.001876
29	20.680300	0.015224	20.176208	0.039228	0.001897
30	20.671252	0.015655	20.167159	0.039659	0.001919
31	20.661844	0.016103	20.157750	0.040107	0.001941
32	20.652065	0.016568	20.147971	0.040573	0.001965
33	20.641906	0.017052	20.137812	0.041057	0.001989
34	20.631357	0.017554	20.127262	0.041559	0.002014
35	20.620405	0.018076	20.116309	0.042081	0.002041
36	20.609038	0.018617	20.104941	0.042622	0.002068
37	20.597240	0.019179	20.093143	0.043184	0.002097
38	20.584995	0.019762	20.080897	0.043767	0.002126
39	20.572284	0.020367	20.068186	0.044372	0.002157
40	20.559087	0.020996	20.054988	0.045001	0.002189
41	20.545382	0.021649	20.041282	0.045653	0.002222
42	20.531143	0.022327	20.027042	0.046331	0.002257
43	20.516346	0.023031	20.012244	0.047036	0.002293
44	20.500963	0.023764	19.996861	0.047769	0.002330
45	20.484965	0.024526	19.980862	0.048530	0.002369
46	20.468322	0.025318	19.964218	0.049323	0.002410
47	20.451002	0.026143	19.946898	0.050148	0.002452
48	20.432972	0.027001	19.928867	0.051006	0.002496
49	20.414196	0.027895	19.910091	0.051900	0.002542
50	20.394639	0.028827	19.890533	0.052832	0.002590
51	20.374262	0.029797	19.870156	0.053802	0.002641
52	20.353025	0.030808	19.848917	0.054813	0.002693
53	20.330885	0.031863	19.826777	0.055868	0.002748
54	20.307799	0.032962	19.803691	0.056967	0.002805
55	20.283721	0.034109	19.779612	0.058114	0.002865
56	20.258603	0.035305	19.754493	0.059310	0.002928
57	20.232394	0.036553	19.728283	0.060558	0.002993
58	20.205042	0.037855	19.700930	0.061860	0.003062
59	20.176492	0.039215	19.672379	0.063220	0.003133
60	20.146686	0.040634	19.642573	0.064639	0.003208
61	20.115565	0.042116	19.611451	0.066121	0.003287
62	20.083065	0.043664	19.578950	0.067669	0.003369
63	20.049122	0.045280	19.545006	0.069285	0.003456
64	20.013667	0.046968	19.509551	0.070974	0.003546
65	19.976630	0.048732	19.472512	0.072738	0.003641
66	19.937936	0.050574	19.433817	0.074580	0.003741

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67	19.897507	0.052500	19.393388	0.076505	0.003845
68	19.855264	0.054511	19.351144	0.078517	0.003954
69	19.811123	0.056613	19.307002	0.080619	0.004069
70	19.764997	0.058810	19.260875	0.082815	0.004190
71	19.716795	0.061105	19.212672	0.085111	0.004317
72	19.666424	0.063504	19.162299	0.087510	0.004450
73	19.613786	0.066010	19.109659	0.090016	0.004589
74	19.558779	0.068630	19.054651	0.092636	0.004736
75	19.501299	0.071367	18.997170	0.095373	0.004891
76	19.441237	0.074227	18.937107	0.098233	0.005053
77	19.378481	0.077215	18.874349	0.101221	0.005223
78	19.312915	0.080337	18.808781	0.104344	0.005403
79	19.244418	0.083599	18.740283	0.107606	0.005592
80	19.172867	0.087006	18.668730	0.111013	0.005790
81	19.098135	0.090565	18.593995	0.114572	0.005999
82	19.020090	0.094281	18.515948	0.118288	0.006219
83	18.938597	0.098162	18.434453	0.122169	0.006451
84	18.853519	0.102213	18.349372	0.126220	0.006695
85	18.764713	0.106442	18.260563	0.130449	0.006952
86	18.672035	0.110855	18.167882	0.134863	0.007223
87	18.575336	0.115460	18.071180	0.139468	0.007508
88	18.474467	0.120263	17.970307	0.144271	0.007809
89	18.369274	0.125273	17.865110	0.149280	0.008127
90	18.259601	0.130495	17.755434	0.154503	0.008461
91	18.145293	0.135938	17.641120	0.159947	0.008815
92	18.026190	0.141610	17.522013	0.165618	0.009188
93	17.902134	0.147517	17.397951	0.171526	0.009581
94	17.772965	0.153668	17.268775	0.177677	0.009997
95	17.638524	0.160070	17.134328	0.184080	0.010436
96	17.498654	0.166731	16.994450	0.190740	0.010900
97	17.353198	0.173657	16.848986	0.197667	0.011391
98	17.202004	0.180857	16.697783	0.204868	0.011910
99	17.044921	0.188337	16.540690	0.212348	0.012458
100	16.881805	0.196105	16.377563	0.220116	0.013039
101	16.712517	0.204166	16.208262	0.228178	0.013653
102	16.536924	0.212527	16.032656	0.236540	0.014304
103	16.354904	0.221195	15.850621	0.245209	0.014993
104	16.166341	0.230174	15.662041	0.254189	0.015723
105	15.971133	0.239470	15.466816	0.263485	0.016498
106	15.769191	0.249086	15.264853	0.273102	0.017319
107	15.560437	0.259027	15.056077	0.283044	0.018190
108	15.344813	0.269295	14.840428	0.293313	0.019115
109	15.122276	0.279892	14.617864	0.303911	0.020097
110	14.892804	0.290819	14.388362	0.314840	0.021140

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111	14.656397	0.302076	14.151922	0.326099	0.022250
112	14.413078	0.313663	13.908566	0.337687	0.023429
113	14.162895	0.325576	13.658342	0.349603	0.024684
114	13.905923	0.337813	13.401325	0.361842	0.026021
115	13.642268	0.350368	13.137619	0.374399	0.027444
116	13.372065	0.363235	12.867360	0.387269	0.028961
117	13.095482	0.376406	12.590716	0.400442	0.030579
118	12.812722	0.389870	12.307888	0.413910	0.032305

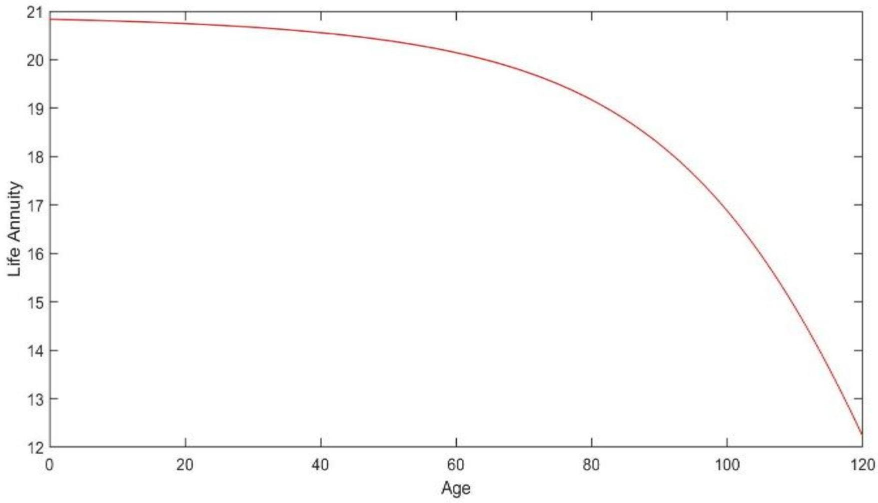

Figure 1: Extended Makeham's Life Annuity Function

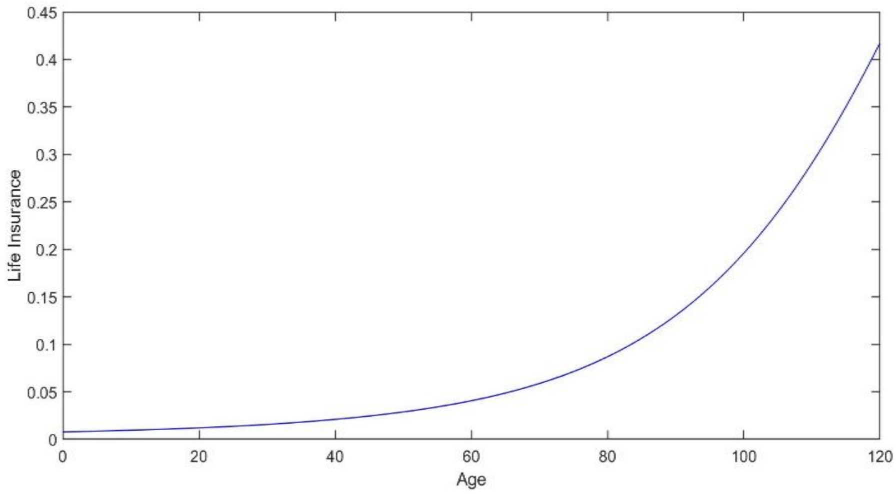

Figure 2: Extended Makeham's Life Insurance Function

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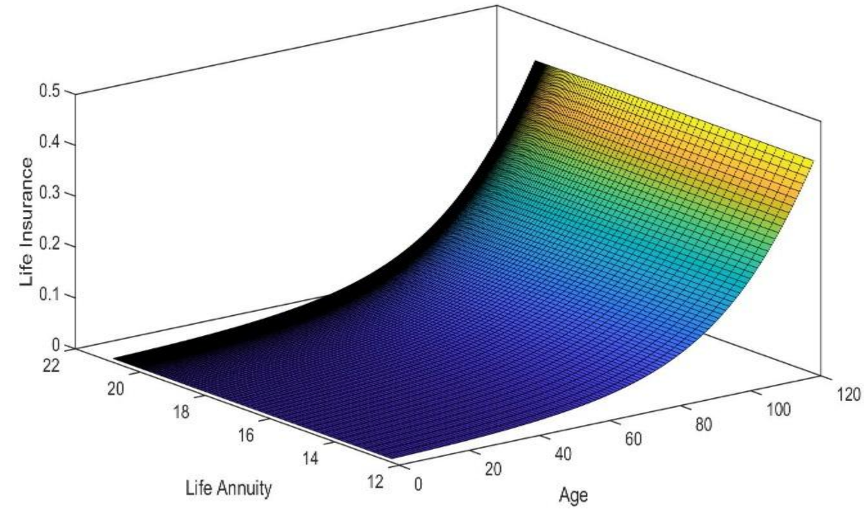

Figure 3: Extended Makeham's Surface Plot for Life Annuity and Life Insurance Function

4.0 Discussion of Results

Taking the limits in (157) yields

$$\lim_{m\to\infty} \ddot{a}_y^{(m)} = \lim_{m\to\infty}\left\{\begin{array}{l} \ddot{a}_y + \dfrac{1}{720m^4}\left[\begin{array}{l} 2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\} \\ -\left(\delta+\mu_y\right)^3 \\ +\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\} \\ -\left\{\left(\log_e C\right)^2 BC^y\right\} \end{array}\right] \\ -\dfrac{1}{720}\left\{\begin{array}{l} 2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3 \\ +\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\} \end{array}\right\} \\ +\dfrac{1}{2}\left(\dfrac{1}{m}-1\right)+\dfrac{1}{12}\left\{\delta+\mu_y\right\}\left(\dfrac{1}{m^2}-1\right) \end{array}\right\} \tag{159}$$

When the number of times life annuity benefits are payable to the annuitant is indefinite such as weekly or hourly, then taking the limit in (159) as $m \to \infty$,

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$$\bar{a}_y = \left\{ \begin{array}{l} \sum_{j=0}^{\Omega-x-1} e^{-\delta j}\left(S^j W^{j^2+2yj} g^{C^{y+j}-C^y}\right) \\ -\frac{1}{720}\left\{ \begin{array}{l} 2\left(\delta+\mu_y\right)\left\{H+\left(\log_e C\right)BC^y\right\}-\left(\delta+\mu_y\right)^3 \\ +\left\{H+\left(\log_e C\right)BC^y\right\}\left\{\delta+\mu_y\right\}-\left\{\left(\log_e C\right)^2 BC^y\right\} \end{array} \right\} \\ -\frac{1}{2}-\frac{1}{12}\left\{\delta+\mu_y\right\} \end{array} \right\} \tag{160}$$

However, if the scheme is incepted on a discrete basis, for a term m years, the life annuity purchased for m years and payable at the beginning of the insurance period is defined as

$$\ddot{a}_{x:\overline{m}|} = \sum_{k=0}^{m-1}\left(1+i\right)^{-k} {}_k p_x \tag{161}$$

Following Ogungbenle, Sirisena, Ukwu, and Adeyele (2025b), the parameters of the generalised Makeham's function were estimated through the novel method of successive differencing as

$$\begin{aligned}&\left(A,B,H,C\right)\\ &=\left(9.99752\times10^{-05},3.93158\times10^{-08},7.48763\times10^{-06},1.109423265\right)\end{aligned} \tag{162}$$

Each column in Table 1 describes the actuarial present value of a life table function of different payment period. At age 0, the discrete life annuity is $\ddot{a}_0 = 20.837337$, but at age 118, it declines to $\ddot{a}_{118} = 12.812722$ while $\bar{a}_0 = 20.333263$ and $\bar{a}_{118} = 12.307888$. As a life advances in age, the expected life time falls and hence the number payments expected from an annuity contract becomes the smaller. At age 0, $A_0 = 0.007746$ and at 118, $A_{118} = 0.389870$, $\bar{A}_0 = 0.031749$ and $\bar{A}_{118} = 0.413910$. With age, mortality rate increases, so the present value of a 1-unit benefit payable at death increases. The result in Table 1 that $\ddot{a}_x > \bar{a}_x$ is analytically proved below

Since T_x defines the future life time of (x), and $\delta = \ln(1+i)$. A continuous whole life annuity pays at the rate of 1 continuously contingent on survival and hence

$$\bar{a}_x = E\left(\int_0^{T_x} e^{-\delta\xi} d\xi\right) \tag{163}$$

Interchange the expectation operator and integral using Fubini's theorem,

$$\bar{a}_x = \int_0^{\infty} e^{-\delta\xi} P\left(T_x > \xi\right) d\xi = \int_0^{\infty} e^{-\delta\xi}\left({}_\xi p_x\right) d\xi \tag{164}$$

A discrete whole life annuity due pays at the beginning of each year provided the life survives and hence

$$\ddot{a}_x = \mathbf{E}\left(\sum_{k=0}^{[T_x]}\left(1+i\right)^{-k}\right) \tag{165}$$

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$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k P\left(T_x > k\right) = \sum_{k=0}^{\infty} v^k \left({}_k p_x \right) \tag{166}$$

$$\overline{a}_x = \sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-\delta\xi} \left({}_\xi p_x \right) d\xi \tag{167}$$

In the kth year, $k \le \xi < k+1$, the consistency principle means

$${}_\xi p_x = \left({}_k p_x \right)\left({}_{\xi-k} p_{x+k} \right) = \frac{l_{x+k}}{l_x} \times \frac{l_{x+\xi}}{l_{x+k}} \tag{168}$$

$$\overline{a}_x = \sum_{k=0}^{\infty} \left({}_k p_x \right) \int_0^1 e^{-\delta(k+s)} \left({}_s p_{x+k} \right) ds \tag{169}$$

$$\overline{a}_x = \sum_{k=0}^{\infty} \left({}_k p_x \right) e^{-\delta k} \int_0^1 e^{-\delta s} \left({}_s p_{x+k} \right) ds = \sum_{k=0}^{\infty} \left({}_k p_x \right) \left(e^{-\delta} \right)^k \int_0^1 e^{-\delta s} \left({}_s p_{x+k} \right) ds \tag{169a}$$

$$\overline{a}_x = \sum_{k=0}^{\infty} \left({}_k p_x \right) v^k \int_0^1 e^{-\delta s} \left({}_s p_{x+k} \right) ds \tag{169b}$$

$$I = \int_0^1 e^{-\delta s} \left({}_s p_{x+k} \right) ds \tag{169c}$$

$e^{-\delta s} > 0$ when $s > 0$ since $\delta > 0$. Furthermore, ${}_s p_{x+k} \le 1$. But ${}_s p_{x+k} < 1$ unless mortality is zero and hence,

$$\int_0^1 e^{-\delta s} \left({}_s p_{x+k} \right) ds < \int_0^1 1 ds = 1 \tag{170}$$

Since every term of the continuous annuity is strictly less than the corresponding discrete term for positive interest and non-zero mortality summing over all k gives $\ddot{a}_x > \overline{a}_x$ for $i > 0$. Discrete annuity due pays at the commencement of each year such that earlier payments result in less discounting. However, continuous annuity pays continuously throughout the year and such payment timing is later and incorporates more discounting. Therefore, at the same interest rate and mortality, the actuarial present value of the discrete annuity due must be greater. Similarly, the result in Table 1 that $\overline{A}_x > A_x$ is due to the arguments that follow. The continuous whole life insurance (pays 1 at the moment of death) has the actuarial present value

$$\overline{A}_x = E\left(e^{-\delta T_x}\right) = \int_0^{\infty} e^{-\delta\xi} \left({}_\xi p_x \right) \mu_{x+\xi} d\xi \tag{171}$$

The discrete whole life insurance (pays 1 at the end of the year of death) has actuarial present value

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \left({}_k p_x \right) q_{x+k} \tag{172}$$

Implicitly, the continuous benefit is paid earlier (on average), and thus has a larger present value when discounted. The probability that death occurs in the year $\left[k, k+1\right)$ is

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$$\mathbf{P}\left(k \le T_x < k+1\right) = \left[{}_k p_x - {}_{k+1} p_x\right] = {}_k\, p_x \times q_{x+k} \tag{173}$$

$$A_x = \sum_{k=0}^{\infty} e^{-\delta(k+1)} \mathbf{P}\left(k \le T_x < k+1\right) \tag{174}$$

$$A_x = \sum_{k=0}^{\infty} e^{-\delta(k+1)} \int_k^{k+1} f_{T_x}\left(\xi\right) d\xi \tag{175}$$

$$f_{T_x}\left(\xi\right) = {}_\xi\, p_x \mu_{x+\xi} \tag{176}$$

$$\overline{A}_x = \int_0^{\infty} e^{-\delta\xi} f_{T_x}\left(\xi\right) d\xi \tag{177}$$

this integral can be partitioned into yearly pieces

$$\overline{A}_x = \sum_{k=0}^{\infty} \left(\int_k^{k+1} e^{-\delta\xi} f_{T_x}\left(\xi\right) d\xi \right) \tag{178}$$

Continuous component

$$\mathbf{C}\left(k,\delta\right) = \int_k^{k+1} e^{-\delta\xi} f_{T_x}\left(\xi\right) d\xi \tag{179}$$

$$\mathbf{D}\left(k,\delta\right) = \int_k^{k+1} e^{-\delta(k+1)} f_{T_x}\left(\xi\right) d\xi \tag{180}$$

$$\overline{A}_x = \sum_{k=0}^{\infty} \mathbf{C}\left(k,\delta\right) \tag{180a}$$

$$A_x = \sum_{k=0}^{\infty} \mathbf{D}\left(k,\delta\right) \tag{180b}$$

$$\text{within } \xi \in \left[k, k+1\right) \tag{180c}$$

$$e^{-\delta k} > e^{-\delta(k+1)} \tag{180d}$$

Therefore

$$f_{T_x}\left(\xi\right) e^{-\delta k} > f_{T_x}\left(\xi\right) e^{-\delta(k+1)} \tag{180e}$$

Integrating both sides over $\xi \in \left[k, k+1\right)$

$$\int_k^{k+1} f_{T_x}\left(\xi\right) e^{-\delta k} d\xi > \int_k^{k+1} f_{T_x}\left(\xi\right) e^{-\delta(k+1)} d\xi \tag{180f}$$

$\mathbf{C}\left(k,\delta\right) > \mathbf{D}\left(k,\delta\right)$ for all $k \ge 0$

Summing across all years $k = 0,1,2,3,...$

$$\overline{A}_x = \sum_{k=0}^{\infty} \mathbf{C}\left(k,\delta\right) > \sum_{k=0}^{\infty} \mathbf{D}\left(k,\delta\right) = A_x \tag{180g}$$

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The strict inequality occurs because for any k unless the mortality density $f_{T_x}(\xi)$ has all its mass points at $\xi = k+1$ which is impossible for a continuous lifetime, there exists a subset of $[k, k+1)$ with positive probability where payment is effected earlier than year-end, producing a strictly greater discounted value. Thus, the result holds for any strictly positive δ and any valid mortality law with continuous density function. From Table 1, at each age, $\ddot{a}_x > \bar{a}_x$, since annuity due payments are disbursed at the beginning of each period and continuously paid annuities are effected subsequently on average, the discrete annuity due has a slightly higher value as a result of the timing advantage. In the relation $\bar{A}_x > A_x$,continuous insurance pays immediately at the moment of death rather than end of year, so the expected value is higher (there's less discounting). Numerically, at age 0, $\ddot{a}_0 - \bar{a}_0 = 0.504074$, and $\bar{A}_0 - A_0 = 0.024003$. This deviation seems small but consistent, exhibiting the payment timing differential between discrete and continuous functions.

At young ages, life annuity values decrease very slowly while the life insurance values increase steadily. At advanced ages, both trajectories accelerate where life annuities decrease faster while insurances grow faster. This exhibits the nonlinear behaviour of the mortality function as the instantaneous mortality increases approximately exponentially with age in line with the Gompertz's function, leading to a steep decline in survival-weighted annuity actuarial present values and a faster accumulation of mortality-weighted insurance actuarial present values. Consequently, the curvature in the annuity trajectories impact a direct numerical contribution from the underlying mortality function. The Figures 1 and 2 clearly demonstrate the duality behaviour between survival and death benefits. However, Figure 3 describe the surface plot of the life annuity, life insurance functions and age. The actuarial identity $A_x + d\ddot{a}_x = 1$ expresses that the total value of the payment stream over a life through annuities or insurance adds up to the present value of certainty as table 1 numerically satisfies the identities $A_x \equiv 1 - d\ddot{a}_x$ and $\bar{A}_x + \delta\bar{a}_x = 1$. This confirms the internal actuarial consistency of the table. The magnitude of annuity values $\left(\ddot{a}_0, \bar{a}_0\right)$ in Table 1 confirms a low to moderate discount rate, probably in the neighbourhood of 5%. This suggests that the model accounts for moderate interest rate, resulting in high annuity factors and low insurance factors. If interest rates were bigger, the life annuity values would decrease faster; if mortality were higher, the life insurance values would increase more rapidly. Hence, the life annuity incorporates implicit mortality-interest equilibrium representative of populations in the post Makeham dynamics. The gaps between discrete and continuous function is narrow across all ages, confirming internal consistency and actuarial accuracy. Analytically, as the payment frequency increases (from annual to continuous), the actuarial present value converges. Table 1 demonstrates this convergence behavior with impressive smoothness, this is an indication that the underlying mortality and interest assumptions are stable and continuous.

The implications for the actuarial present value of annuity are that (i) at younger ages, annuities are more expensive to purchase because payments are expected for longer period (ii) As age advances, annuity cost declines but consistent with the reduction in the expected duration and growing mortality rates (iii) the differential between discrete and continuous annuities describes the timing of premium payment $P\left(\bar{A}_x\right)$ in Table 1which represents a key parameter in pension payout design. However**,** the implications for life insurance present values are that (i) insurance is steadily more expensive with age. (ii) the nonlinear increase confirms exponential mortality, consistent with Gompertz parameters. (iii) continuous life insurance's higher value incorporates immediate payment which is important in reserving and solvency requirements. Table 1 can be employed to extract implied survival curves and to identify mortality gradients. The smoothness and slow decay of annuity values confirm high survival probabilities at young and middle-ages and only a steep decline at advanced ages-characteristic of current mortality structures.

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Furthermore, the decreasing behaviour of life annuity $\bar{a}_x$ incepted on a continuous basis as exhibited in Tables 1 is also due to the analytical arguments observed below

$$\bar{a}_x = \int_0^\infty e^{-\delta\xi}\left({}_\xi p_x\right)d\xi \tag{181}$$

Replace x by $x+\xi$ in (214) yields

$$\bar{a}_{x+\xi} = \int_0^\infty e^{-\delta\tau}\left({}_\tau p_{x+\xi}\right)d\tau \tag{182}$$

Take the derivative of (215) with respect to the arbitrary time ξ to get

$$\frac{\partial}{\partial\xi}\bar{a}_{x+\xi} = \int_0^\infty \frac{\partial}{\partial\xi}e^{-\delta\tau}\left({}_\tau p_{x+\xi}\right)d\tau = \int_0^\infty e^{-\delta\tau}\frac{\partial}{\partial\xi}\left({}_\tau p_{x+\xi}\right)d\tau \tag{183}$$

The derivative under the integral implies

$$\frac{\partial}{\partial\xi}\left({}_\tau p_{x+\xi}\right) = \frac{\partial}{\partial\xi}\left(\frac{l_{x+\xi+\tau}}{l_{x+\xi}}\right) = \frac{l_{x+\xi}\left(\frac{\partial}{\partial\xi}l_{x+\xi+\tau}\right) - l_{x+\xi+\tau}\left(\frac{\partial}{\partial\xi}l_{x+\xi}\right)}{\left(l_{x+\xi}\right)^2} \tag{184}$$

The force of mortality in equations (8) and (9) implies that

$$\frac{dl_{x+\xi+\tau}}{d\xi} = -\mu_{x+\xi+\tau}\times l_{x+\xi+\tau} \tag{185}$$

$$\frac{\partial}{\partial\xi}\left({}_\tau p_{x+\xi}\right) = \frac{l_{x+\xi}\left(-\mu_{x+\xi+\tau}l_{x+\xi+\tau}\right) - l_{x+\xi+\tau}\left(-\mu_{x+\xi}l_{x+\xi}\right)}{\left(l_{x+\xi}\right)^2} \tag{186}$$

$$\frac{\partial}{\partial\xi}\left({}_\tau p_{x+\xi}\right) = -\frac{l_{x+\xi+\tau}\times\mu_{x+\xi+\tau}}{l_{x+\xi}} + \frac{l_{x+\xi+\tau}\times\mu_{x+\xi}}{l_{x+\xi}} = -{}_\tau p_{x+\xi}\mu_{x+\xi+\tau} + {}_\tau p_{x+\xi}\mu_{x+\xi} \tag{187}$$

Substitute (221) in (216) to get

$$\begin{aligned}\frac{\partial}{\partial\xi}\bar{a}_{x+\xi} &= \int_0^\infty e^{-\delta\tau}\left[-{}_\tau p_{x+\xi}\mu_{x+\xi+\tau} + {}_\tau p_{x+\xi}\mu_{x+\xi}\right]d\tau \\ &= \mu_{x+\xi}\int_0^\infty\left[e^{-\delta\tau}{}_\tau p_{x+\xi}\right]d\tau - \int_0^\infty\left[e^{-\delta\tau}{}_\tau p_{x+\xi}\mu_{x+\xi+\tau}\right]d\tau\end{aligned} \tag{188}$$

$$\frac{\partial}{\partial\xi}\bar{a}_{x+\xi} = \mu_{x+\xi}\bar{a}_{x+\xi} - \bar{A}_{x+\xi} \tag{189}$$

Observed in Table 1 that, $\mu_x \ll 1$ and $\bar{A}_x < 1$ and hence results in (222) will less than zero

$$\frac{\partial}{\partial\xi}\bar{a}_{x+\xi} < 0 \tag{190}$$

and hence $\bar{a}_x$ is a decreasing function.

5.0 Conclusions

This paper deals with the actuarial properties and contingency functions of the extended Makeham's mortality model in the context of life insurance benefits. Deploying advanced analytic techniques under closed form expressions were obtained for the benefit function theorem, integrated hazard function, whole life annuity and insurance functions. These results enabled us to construct benefits tables directly from the benefit function theorem without the need to adopt commutation functions. The analytic tractability achieved for life annuity function therefore advances the domain of morality functions available for rigorous, model-based evaluation of life insurance products beyond

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pontoonic arguments. A major contribution of this paper is the observation of a structural duality principle connecting the trajectories of life annuity and life insurance functions generated under this mortality framework. The evolution of this principle highlights a deep coherence within the model representing an advanced innovative framework for closed form actuarial expressions. It also confirms that the previously neglected integrated hazard property has potential practical applications and theoretical depth for life insurance pricing. The findings widen the analytic scope of life contingencies and offer further opportunities in the modelling of longevity risk, actuarial valuation and annuity products.

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